Skew-orthogonal polynomials and random-matrix ensembles

Saugata Ghosh and Akhilesh Pandey
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India
(Received 16 November 2001; published 8 April 2002)

There is considerable interest in understanding the relation between random-matrix ensembles and quantum chaotic systems in the context of the universality of energy-level correlations. In this connection, while Gaussian ensembles of random matrices have been studied extensively, not much is known about ensembles with non-Gaussian weight functions. Dyson has shown that the *n*-level correlation functions can be expressed in terms of a kernel function involving orthogonal and skew-orthogonal polynomials—orthogonal for matrix ensembles with unitary invariance and skew orthogonal for ensembles with orthogonal and symplectic invariances. We have obtained the following results. (1) Skew-orthogonal polynomials of both types are derived for the Jacobi class of weight functions including the limiting cases of associated Laguerre and Hermite (or Gaussian). (2) Matrix-integral representations are given for the general weight functions. (3) Asymptotic forms of the polynomials are obtained rigorously for the Jacobi class and in the form of an ansatz for the general case. (4) For the three types of ensembles, the (asymptotic) *n*-level correlation functions with appropriate scaling are shown to be universal, being independent of the weight function and location in the spectrum, and identical with the well-known Gaussian results. This provides a rigorous justification for the universality of the Gaussian ensemble results observed in quantum chaotic systems. As expected, the level density is not universal.

DOI: 10.1103/PhysRevE.65.046221 PACS number(s): 05.45.Mt

I. INTRODUCTION

Universality of energy-level fluctuations (i.e., deviations from local uniformity) is observed in a wide variety of quantum chaotic systems. Spectra of complex nuclei, atoms, molecules, disordered mesoscopic systems, and microwave cavities provide experimental verification of the universality. Numerical and semiclassical studies of chaotic systems with few degrees of freedom as well as studies of zeros of the Riemann zeta function confirm the same. The observed fluctuations are in close agreement with the analytic results of the Gaussian ensembles of random matrices [1–6].

The Gaussian ensembles have been studied as mathematical tools rather than as physical models. For example, the predicted level density, viz., the ''semicircle,'' does not correspond to any known physical system. It is therefore of interest [7] to study matrix ensembles with non-Gaussian weight functions, which give very different level densities. Our main aim in this paper is to establish rigorously the universal behavior of energy-level fluctuations for a wide class of matrix ensembles. This would provide a firm justification for the universality found in physical systems. The second aim of this paper is to develop the theory of skew-orthogonal polynomials, necessary, as shown by Dyson [7,8], for such a study. A brief account of this work has been given elsewhere [9].

As in the Gaussian case, we follow the threefold classification to take account of the fundamental symmetries of the system, and study ensembles of three types of matrices, viz., real symmetric, complex Hermitian, and self-dual quaternion real [1]. The three types of ensembles will be characterized by the parameter β, with values 1,2,4, respectively, denoting the number of real ''sites'' in each off-diagonal matrix element. The ensembles will be invariant under orthogonal, unitary, and symplectic transformations respectively in the three cases, being therefore referred to as orthogonal, unitary and symplectic ensembles—OE, UE, and SE in short. A further mathematical requirement that the matrix weight function be separable in eigenvalues would complete the definition of the ensembles that we consider. It turns out that, with the Gaussian weight functions, the distinct matrix elements are statistically independent, but not so with the other weight functions. It is then possible to deal with matrix ensembles having a specified level density [7,10]. Such ensembles will be of direct relevance to many-body systems such as complex nuclei [2] and mesoscopic systems [4,5].

Dyson [7] has shown that the eigenvalue-density correlation functions can be written in terms of standard orthogonal polynomials in the unitary case, and in terms of certain skew-orthogonal polynomials in the orthogonal and symplectic cases. A major part of the paper will be concerned with the derivation of the skew-orthogonal polynomials for the Jacobi class of weight functions, giving thereby generalizations of some initial results of Mehta [8]. Their asymptotic forms will then establish the universality for the Jacobi ensembles, as reported earlier in the unitary case by Fox and Kahn [11]. For more general weight functions, we shall derive matrix-integral representations of the polynomials, which are more amenable to asymptotic studies [12,13]. Finally, we shall propose an ansatz for the asymptotic forms of the polynomials, as in Ref. [14], whereby universality will be established more generally. It should be emphasized that the three types of the universal energy-level fluctuations will not only be independent of the weight function, but also independent of the location in the spectrum (''stationarity'' [15]). For other recent work on non-Gaussian matrix ensembles, we refer to Refs. [4,5] and references therein.

In Sec. II, we give the basic definitions along with a discussion of level densities in the general ensembles and a brief review of the (asymptotic) correlation functions for the Gaussian ensembles. Sections III, IV, V, and VI are concerned with the correlation functions for the Jacobi ensembles (including the limiting cases of associated Laguerre and Hermite or Gaussian), respectively, for UE, OE (even dimensions), OE (odd dimensions), and SE. Section VII gives the matrix-integral representations of the polynomials and the ansatz for the asymptotic forms. Our results are summarized in the concluding section.

II. LEVEL DENSITY AND CORRELATION FUNCTIONS

We consider ensembles of N-dimensional Hermitian matrices (H) with the probability distribution [7]

$$P_{\beta,N}(H)dH = C_{\beta,N}\exp[-tr u(H)]dH, \tag{2.1}$$

where the parameter β, defined in Sec. I, denotes whether H is real symmetric, complex Hermitian or quaternion-real self-dual, dH being the infinitesimal volume element in the space of these matrices. The matrix function $u(H)$ is defined by the power expansion of the function $u(z)$, and $C_{\beta,N}$ is fixed by the normalization condition. The Gaussian ensembles—GOE, GUE, and GSE, respectively, for $\beta = 1,2,4$—are obtained with $u(z) = z^2/2v^2$, v being a scale parameter. From the invariance of the ensembles, the joint-probability density of the eigenvalues $(x_1, x_2, \ldots, x_N)$ is obtained easily [7],

$$\mathcal{P}_{\beta,N}(x_1, \ldots, x_N) = c_{\beta,N}|\Delta_N(x_1, \ldots, x_N)|^{\beta}\prod_{j=1}^{N} w(x_j), \tag{2.2}$$

where $c_{\beta,N}$ is another normalization constant,

$$\Delta_N(x_1, \ldots, x_N) = \prod_{j<k}(x_j - x_k) \tag{2.3}$$

is the Vandermonde determinant, and

$$w(x) = \exp[-u(x)] \tag{2.4}$$

is the weight function. It is sometimes helpful to think of the ensemble distribution (2.1) as coming from an information theory approach [10] or as the equilibrium density of a Brownian-motion process [7]; in the former case the function u (and hence the weight function w) comes from the constraints on the ensemble, while in the latter case u plays the role of a potential.

The n-eigenvalue or n-level density correlation function [1,7] R_n (for $n = 1,2,3, \ldots, N$) is defined by

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!}\int dx_{n+1}\ldots\int dx_N \times \mathcal{P}(x_1, \ldots, x_N), \tag{2.5}$$

and gives the probability density of finding n eigenvalues at $x_1, x_2, \ldots x_n$, irrespective of the positions of the remaining eigenvalues. $R_1(x)$ is the level density, with

$$D(x) = [R_1(x)]^{-1} \tag{2.6}$$

being the mean level spacing at x for large N. To describe the energy level fluctuations in the neighborhood of x, we first "unfold" the spectrum locally by

$$x_j = x + r_j D(x), \quad j = 1,2, \ldots, n. \tag{2.7}$$

Then

$$\mathbf{R}_n(r_1, r_2, \ldots, r_n; x) = \lim_{N\to\infty}(D(x))^n R_n(x_1, \ldots, x_n) \tag{2.8}$$

is the n-level correlation function for the unfolded spectra. In the cases we consider in this paper, $\mathbf{R}_n$ will be independent of x, being thus stationary.

The Jacobi weight function, the main topic of concern in Secs. III–VI, is given by

$$\begin{aligned} w_{a,b} &= (1-x)^a(1+x)^b, \quad |x|<1, \\ &= 0, \quad |x|>1, \end{aligned} \tag{2.9}$$

where $a > -1$, $b > -1$. The classical (orthogonal) polynomials [16] derive from the Jacobi weight function. Thus, for $a = b$, we have the Gegenbauer, which includes Legendre for $a = b = 0$ and the two Chebyshev for $a = b = \pm\frac{1}{2}$. Also by rescaling x and then letting $a = b \to \infty$, we get the Hermite or Gaussian weight function,

$$w_G(x) = \exp(-\beta x^2/2), \tag{2.10}$$

where β has been inserted in the exponent to make correspondence with the results of Ref. [1]. Moreover, by shifting the origin, then rescaling x, w, and letting $b \to \infty$ we get the associated Laguerre weight function,

$$\begin{aligned} w_a(x) &= x^a e^{-x}, \quad x>0, \\ &= 0, \quad x<0, \end{aligned} \tag{2.11}$$

where $a > -1$. Besides the Jacobi, we shall also consider [14] in Sec. VII the case when the "potential" $u(x) = -\ln w(x)$ is a low-order polynomial.

We now give a general procedure for deriving the level density for large N. Since

$$\frac{\partial\mathcal{P}(x_1, \ldots, x_N)}{\partial x_1} = \left(\beta\sum_{j\neq 1}\frac{1}{x_1 - x_j} + \frac{w'(x_1)}{w(x_1)}\right)\mathcal{P}(x_1, \ldots, x_N), \tag{2.12}$$

we find from Eq. (2.5) an exact hierarchic set of relations linking R_n to R_{n+1} [17–19]. For $n = 1$, this gives

$$\frac{\partial R_1(x)}{\partial x} = \beta\int\frac{R_2(x,y)}{x-y}dy + \frac{w'(x)}{w(x)}R_1(x). \tag{2.13}$$

For large N, the integral on the right-hand side can be replaced by a principal-value integral involving $R_2(x,y) \approx R_1(x)R_1(y)$; moreover $\partial R_1/\partial x$ can be dropped. Both these approximations can be rigorously justified from the behavior of R_1 and R_2 for large N. We thus find [7,5,17]

$$\beta R_1(x)\int \frac{R_1(y)}{x-y}dy+\frac{w'(x)}{w(x)}R_1(x)=0, \quad (2.14)$$

a result that could also be derived directly by maximizing $\ln\mathcal{P}$. Note that Eq. (2.14) is valid when $w\neq 0$. Moreover, R_1 is zero when $w=0$ but it can also be zero elsewhere as, for example, in the Gaussian and Laguerre cases below. We solve below the integral equation (2.14), using the resolvent [20]

$$G(z)=\int \frac{R_1(y)}{z-y}dy, \quad (2.15)$$

which satisfies

$$G(x+i0)=\int \frac{R_1(y)}{x-y}dy-i\pi R_1(x). \quad (2.16)$$

For the Jacobi weight function (2.9), one needs to consider carefully the singularities of w'/w at $x=\pm 1$. Since $R_1(x)=0$ for $|x|>1$, and $O(N)$ for $|x|\leqslant 1$, we find from Eq. (2.14), after multiplication by $(1-x^2)/(z-x)$ and integration over x, that

$$\int_{-1}^{1}dx\frac{(1-x^2)R_1(x)}{z-x}\int_{-1}^{1}dy\frac{R_1(y)}{x-y}=0, \quad (2.17)$$

terms ignored being of lower order in N. The expression on the left can be written as

$$\frac{1}{2}\int\int_{-1}^{1}dxdy\frac{R_1(x)R_1(y)}{x-y}\left(\frac{1-x^2}{z-x}-\frac{1-y^2}{z-y}\right)$$
$$=\frac{1}{2}\int\int_{-1}^{1}dxdy\frac{R_1(x)R_1(y)}{(z-x)(z-y)}\{1-z(x+y)+xy\}$$
$$=\frac{1}{2}(1-z^2)G^2+\frac{N^2}{2}. \quad (2.18)$$

Thus, since $G(z)\approx N/z$ for large $|z|$, we get

$$G(z)=\frac{N}{\sqrt{z^2-1}}, \quad (2.19)$$

and

$$R_1(x)=\frac{N}{\pi\sqrt{1-x^2}}, \quad |x|<1,$$
$$=0, \quad |x|>1, \quad (2.20)$$

the result being the same for all finite values of the parameters a,b. Note that the level density becomes indefinitely large at the end points. For the associated Laguerre weight function (2.11), $R_1(x)=0$ for $x<0$, and w'/w has a singularity at $x=0$. In this case we find from Eq. (2.14), after multiplication by $x/(z-x)$, integration over x, and neglecting the (lower order) $aG(z)$ term, that

$$-\int_0^{\infty}\frac{xR_1(x)}{z-x}dx+\beta\int_0^{\infty}dx\frac{xR_1(x)}{z-x}\int_0^{\infty}dy\frac{R_1(y)}{x-y}=0, \quad (2.21)$$

so that

$$\beta zG^2-2zG+2N=0, \quad (2.22)$$

giving

$$G(z)=\frac{1}{\beta}-\frac{1}{\beta}\sqrt{\frac{z-2\beta N}{z}}, \quad (2.23)$$

and

$$R_1(x)=\frac{1}{\pi\beta}\sqrt{\frac{2\beta N-x}{x}}, \quad 2\beta N>x>0,$$
$$=0, \quad x<0 \quad \text{or} \quad x>2\beta N, \quad (2.24)$$

again independent of the parameter a, but different from the Jacobi result (2.20). Finally, in the Gaussian case [20], we get from Eqs. (2.10), (2.14),

$$-\int_{-\infty}^{\infty}\frac{xR_1(x)}{z-x}dx+\int_{-\infty}^{\infty}dx\frac{R_1(x)}{z-x}\int_{-\infty}^{\infty}dy\frac{R_1(y)}{x-y}=0, \quad (2.25)$$

implying

$$G^2-2zG+2N=0. \quad (2.26)$$

Thus

$$G(z)=z-\sqrt{z^2-2N}, \quad (2.27)$$

and

$$R_1(x)=\frac{1}{\pi}\sqrt{2N-x^2}, \quad |x|<\sqrt{2N},$$
$$=0, \quad |x|>\sqrt{2N}, \quad (2.28)$$

the last result being the well-known semicircular density [1]. One can similarly obtain results for $G(z)$ and $R_1(x)$ when $u(x)$ is $O(N)$ and a low-order polynomial; this gives results different from above.

The polynomial methods discussed in Secs. III–VI will confirm the above density results. We shall also find that the unfolded correlation functions $\mathbf{R}_n$ are universal, being independent of the weight function and location in the spectrum. We summarize here the results for $\mathbf{R}_n$, first obtained for the Gaussian and circular ensembles [21,22,7,1]. We have for the three types of ensembles

$$\mathbf{R}_n^{(\beta)}(r_1,\ldots,r_n)=Q\det[\sigma_\beta(r_j-r_k)]_{j,k=1,\ldots,n}$$
$$=\{\det[\sigma_\beta(r_j-r_k)]\}^{1/2}, \quad (2.29)$$

where

$$\sigma_2(r)=\begin{pmatrix} S(r) & 0 \\ 0 & S(r) \end{pmatrix}, \tag{2.30}$$

$$\sigma_1(r)=\begin{pmatrix} S(r) & D(r) \\ I(r)-\epsilon(r) & S(r) \end{pmatrix}, \tag{2.31}$$

$$\sigma_4(r)=\begin{pmatrix} S(2r) & D(2r) \\ I(2r) & S(2r) \end{pmatrix}, \tag{2.32}$$

$$S(r)=\sin(\pi r)/\pi r, \tag{2.33}$$

$$D(r)=dS(r)/dr, \tag{2.34}$$

$$I(r)=\int_0^r S(r')dr', \tag{2.35}$$

$$\epsilon(r)=r/2|r|. \tag{2.36}$$

The quaternion determinant (Q det) in Eq. (2.29) has a determinantlike expansion in terms of the quaternion matrix elements $\sigma_\beta(r_j-r_k)$ of the n-dimensional "self-dual" quaternion matrix. In fact, it is square root of the $(2n)$-dimensional ordinary determinant, as given in the last step of Eq. (2.29); for example, for $\beta=2$, $\mathbf{R}_n$ is simply $\det[S(r_j-r_k)]$. The function $S(r)$, from which all other functions above derive, will be seen in the following sections as the asymptotic form of a kernel function involving orthogonal polynomials for $\beta=2$ and skew-orthogonal polynomials for $\beta=1,4$. The universality of $\mathbf{R}_n$ will follow from that of the asymptotic kernel function $S(r)$.

III. JACOBI UNITARY ENSEMBLES

The unitary ensemble is easiest to study because the correlation functions can be written in terms of the orthogonal polynomials. Fox and Kahn [11] showed the universality for the Jacobi and Laguerre weight functions in the region of the spectrum where the level density is "flat." We follow here the method of Ref. [15] to extend the universality to all regions where the density is finite and nonzero. Some of the methods used here will also be helpful in the following sections.

The correlation functions in the unitary case ($\beta=2$) can be written as [7]

$$R_n(x_1,\ldots,x_n)=\det[S_N^{(2)}(x_j,x_k)]_{j,k=1,\ldots,n}, \tag{3.1}$$

where $S_N^{(2)}(x,y)$ is the kernel function

$$S_N^{(2)}(x,y)=w(x)\sum_{j=0}^{N-1}(h_j)^{-1}p_j(x)p_j(y). \tag{3.2}$$

The $p_j(x)$ are orthogonal polynomials of order j defined with respect to the weight function $w(x)$ by

$$\int p_j(x)p_k(x)w(x)dx=h_j\delta_{jk}, \tag{3.3}$$

h_j being the normalization constant. Using the Christoffel-Darboux formula [16], we can perform the sum in Eq. (3.2) to obtain

$$S_N^{(2)}(x,y)=\frac{w(x)}{h_{N-1}}\frac{k_{N-1}}{k_N}\left(\frac{p_N(x)p_{N-1}(y)-p_N(y)p_{N-1}(x)}{x-y}\right), \tag{3.4}$$

where k_N is the coefficient of x^N in $p_N(x)$. This form is useful in deriving the large-N behavior of K_N. The level density is given by

$$R_1(x)=S_N^{(2)}(x,x), \tag{3.5}$$

and, therefore, from Eq. (2.8), $S_N^{(2)}(x,y)/S_N^{(2)}(x,x)$ for large N will be required to calculate the unfolded correlation function $\mathbf{R}_n^{(2)}$.

For the Jacobi weight function (2.9), the $p_j(x)$ are the Jacobi polynomials $P_j^{a,b}(x)$ [16], with

$$h_j^{a,b}=\frac{2^{a+b+1}}{(2j+a+b+1)}\frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma(j+1)\Gamma(j+a+b+1)}, \tag{3.6}$$

$$k_j^{a,b}=\frac{1}{2^j}\binom{2j+a+b}{j}. \tag{3.7}$$

For large j and finite a,b with $x=\cos\theta$ $(0<\theta<\pi)$, we have the asymptotic form

$$(h_j^{a,b})^{-1/2}[w_{a,b}(x)]^{1/2}P_j^{a,b}(x)=\sqrt{\frac{2}{\pi\sin\theta}}\cos\left[\left(j+\frac{a+b+1}{2}\right)\theta-\left(a+\frac{1}{2}\right)\frac{\pi}{2}\right], \tag{3.8}$$

where we have used Ref. [16] and $h_j^{a,b}\simeq 2^{(a+b)}j^{-1}$. Here, and in other asymptotic results below, it should be understood that terms of lower order in j for fixed θ are being ignored. The asymptotic form is valid for all θ except within a range $O(j^{-1})$ of the end points. (Also, in Laguerre and Hermite cases below, the asymptotic polynomials fall off exponentially outside the intervals specified.) Now, substituting Eqs. (3.6)–(3.8) in Eq. (3.4) we find, for large N,

$$S_N^{(2)}(x,y)=\frac{\sin(N\Delta\theta)}{\pi\Delta\theta\sin\theta}=\frac{\sin[N(1-x^2)^{-1/2}\Delta x]}{\pi\Delta x}, \tag{3.9}$$

where we have taken $y\equiv x+\Delta x=\cos(\theta+\Delta\theta)$ with $\Delta\theta=O(1/N)$. Thus, with $\Delta\theta\to 0$, we have the level density

$$R_1(x)=\frac{N}{\pi\sin\theta}=\frac{N}{\pi\sqrt{1-x^2}}, \quad |x|<1, \tag{3.10}$$

as derived earlier (2.20). Moreover,

$$\lim_{N\to\infty}\frac{S_N^{(2)}(x,y)}{S_N^{(2)}(x,x)}=\frac{\sin\pi r}{\pi r}=S(r), \tag{3.11}$$

where $r=\Delta x R_1(x)=-\pi^{-1}N\Delta\theta$. [When $\Delta\theta=O(1)$, the above limit is zero.] Using Eqs. (3.1), (3.10), and (3.11) in Eq. (2.8) we rederive the unfolded correlation function of Eqs. (2.29), (2.30). Note that the final result for $\mathbf{R}_n^{(2)}$ is independent of x, as well as a,b, proving thereby the stationarity as well as universality for this class of weight function.

For the associated Laguerre weight function (2.11), the $p_j(x)$ are the associated Laguerre polynomials $L_j^{(a)}(x)$ [16], with

$$h_j^{(a)}=\frac{\Gamma(j+a+1)}{j!}, \tag{3.12}$$

$$k_j^{(a)}=\frac{(-1)^j}{j!}. \tag{3.13}$$

For large j and finite a with $x=(4j+2a+2)\cos^2\theta$, $0<\theta<\pi/2$, we have [16]

$$(h_j^{(a)})^{-1/2}[w_a(x)]^{1/2}L_j^{(a)}(x) =\frac{(-1)^j}{\sqrt{2\pi j\sin\theta\cos\theta}}\sin\left[[j+(a+1)/2]\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{3.14}$$

where $h_j^{(a)}\simeq j^a$. For given x, θ depends on j; for example, $\theta_j-\theta_{j\mp1}\simeq\pm(2j\tan\theta_j)^{-1}$. Then with $\theta\equiv\theta_j$, we can also write

$$(h_{j\mp1}^{(a)})^{-1/2}[w_a(x)]^{1/2}L_{j\mp1}^{(a)}(x) =\frac{(-1)^{j-1}}{\sqrt{2\pi j\sin\theta\cos\theta}}\sin\left[[j+(a+1)/2]\times(\sin 2\theta-2\theta)\pm2\theta+\frac{3\pi}{4}\right]. \tag{3.15}$$

Using Eqs. (3.14), (3.15) in Eq. (3.4) with $\theta\equiv\theta_N$, we get for large N

$$S_N^{(2)}(x,y)=\frac{\sin(4N\Delta\theta\sin^2\theta)}{8\pi N\Delta\theta\sin\theta\cos\theta} =\frac{\sin\left[\frac{1}{2}(4N-x)^{1/2}x^{-1/2}\Delta x\right]}{\pi\Delta x}, \tag{3.16}$$

where $x=4N\cos^2\theta, y\equiv x+\Delta x=4N\cos^2(\theta+\Delta\theta)$, and $\Delta\theta=O(1/N)$. With $\Delta\theta\to0$, we have the level density

$$R_1(x)=\frac{\tan\theta}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{4N-x}{x}},\quad 0<x<4N, \tag{3.17}$$

as in Eq. (2.24) with $\beta=2$, while, with $r=\Delta x R_1(x)=-\pi^{-1}4N\Delta\theta\sin^2\theta$, we have the $S(r)$ function as in Eq. (3.11). This proves stationarity and universality of $\mathbf{R}_n^{(2)}$ in the associated Laguerre case.

For the Gaussian weight function (2.10) (with $\beta=2$), the $p_j(x)$ are the Hermite polynomials [16] $H_j(x)$, with

$$h_j=\pi^{1/2}2^j j!, \tag{3.18}$$

$$k_j=2^j, \tag{3.19}$$

having the asymptotic form [16]

$$(h_j)^{-1/2}e^{-x^2/2}H_j(x) =\frac{1}{\sqrt{\pi\sin\theta}}\left(\frac{2}{j}\right)^{1/4}\sin\left[(j/2+1/4)(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{3.20}$$

for $x=(2j+1)^{1/2}\cos\theta$, $0<\theta<\pi$. As in the Laguerre case, with $\theta\equiv\theta_j$, we can also write

$$(h_{j\mp1})^{-1/2}e^{-x^2/2}H_{j\mp1}(x)=\frac{1}{\sqrt{\pi\sin\theta}}\left(\frac{2}{j}\right)^{1/4}\sin\left[(j/2+1/4)\times(\sin 2\theta-2\theta)\pm\theta+\frac{3\pi}{4}\right], \tag{3.21}$$

where we have used again $\theta_j-\theta_{j\mp1}\simeq\pm(2j\tan\theta_j)^{-1}$. Using θ and $\theta+\Delta\theta$ for x,y with $j=N$, we find [15]

$$S_N^{(2)}(x,y)=\frac{\sin(2N\Delta\theta\sin^2\theta)}{\pi\sqrt{2N}\Delta\theta\sin\theta}=\frac{\sin[(2N-x^2)^{1/2}\Delta x]}{\pi\Delta x}, \tag{3.22}$$

giving

$$R_1(x)=\frac{\sqrt{2N}}{\pi}\sin\theta=\frac{\sqrt{2N-x^2}}{\pi},\quad |x|<\sqrt{2N}, \tag{3.23}$$

as in Eq. (2.28) and $S(r)$ as in Eq. (3.11) with $r=\Delta x R_1(x)=-\pi^{-1}2N\Delta\theta\sin^2\theta$.

The orthogonal polynomials and their asymptotic forms will be needed to work out the skew-orthogonal polynomials and their asymptotic forms in the following sections. We also remark that, instead of the Christoffel-Darboux summation, we could have directly obtained the asymptotic $S_N^{(2)}(x,y)$ by using the asymptotic forms of the polynomials in Eq. (3.2) and replacing the sum by an integral over j for fixed x,y; this procedure will be useful in the skew-orthogonal cases. Moreover, the asymptotic forms (3.8), (3.14), and (3.20) of the polynomials, when written in terms of R_1, are generalizable; see Sec. VII.

IV. JACOBI ORTHOGONAL ENSEMBLES (EVEN DIMENSION)

Dyson [7,8] has given formal expressions for the correlation functions, analogous to Eqs. (3.1)–(3.3), for the $\beta=1,4$ cases. These are given in terms of skew-orthogonal polynomials. There are three types of skew-orthogonal polynomials needed, corresponding to $\beta=1$ (even-N case),

$\beta=1$ (odd-N case), and $\beta=4$. The universality of correlation functions will follow from the asymptotic properties of these polynomials. We consider in this section the first type for the Jacobi family.

The skew-orthogonal polynomials $q_j(x)$ of order j, appropriate for $\beta=1$ (even-N case), are defined by

$$\int\int q_j(x)q_k(y)w(x)w(y)\epsilon(x-y)dxdy=g_jZ_{jk}, \tag{4.1}$$

where Z is the canonical antisymmetric matrix with matrix elements

$$\begin{aligned} Z_{jk}&=1, \quad j=\text{even}, \quad k=j+1,\\ &=-1, \quad j=\text{odd}, \quad k=j-1,\\ &=0, \quad \text{otherwise}, \end{aligned} \tag{4.2}$$

where $j,k=0,1,2,\ldots$, and g_j is the normalization constant with the property $g_{2m}=g_{2m+1}$. The antisymmetric integral in Eq. (4.1) is nonzero for each pair of polynomials q_{2l},q_{2l+1}, implying thereby separate patterns for even- and odd-order polynomials. Apart from the normalization, the even-order polynomials are unique, but to the odd-order ones, any multiple of the *next* lower even-order polynomial can be added. Unlike the orthogonal polynomials, the q_j do not satisfy any three-term recursion relation. However, a Gram-Schmidt-like procedure can be used to construct the polynomials. We also remark that, in Refs. [7–9], the polynomials used are normalized to unity.

To write Eq. (4.1) and other relations in compact form, we introduce the weighted polynomials $\phi_j(x)$ and their integrals $\psi_j(x)$,

$$\phi_j(x)=w(x)q_j(x)=\frac{d\psi_j(x)}{dx}, \tag{4.3}$$

$$\psi_j(x)=\int\epsilon(x-y)\phi_j(y)dy, \tag{4.4}$$

with ϵ defined in Eq. (2.36). Then Eq. (4.1) is equivalent to

$$\int\phi_j(x)\psi_k(x)dx=g_jZ_{jk}. \tag{4.5}$$

We define the kernels

$$\begin{aligned} S_N^{(1)}(x,y)&=\sum_{j,k=0}^{N-1}(g_j)^{-1}Z_{jk}\phi_j(x)\psi_k(y)\\ &=\sum_{m=0}^{(N/2)-1}(g_{2m})^{-1}[\phi_{2m}(x)\psi_{2m+1}(y)\\ &\quad-\phi_{2m+1}(x)\psi_{2m}(y)], \end{aligned} \tag{4.6}$$

$$S_N^{(1)\dagger}(x,y)=-\sum_{j,k=0}^{N-1}(g_j)^{-1}Z_{jk}\psi_j(x)\phi_k(y)=S_N^{(1)}(y,x), \tag{4.7}$$

$$D_N^{(1)}(x,y)=-\sum_{j,k=0}^{N-1}(g_j)^{-1}Z_{jk}\phi_j(x)\phi_k(y)=-\frac{\partial S_N^{(1)}(x,y)}{\partial y}, \tag{4.8}$$

$$\begin{aligned} I_N^{(1)}(x,y)&=\sum_{j,k=0}^{N-1}(g_j)^{-1}Z_{jk}\psi_j(x)\psi_k(y)\\ &=\int\epsilon(x-z)S_N^{(1)}(z,y)dz, \end{aligned} \tag{4.9}$$

$$\epsilon(x,y)=\epsilon(x-y). \tag{4.10}$$

In terms of the quaternion kernel,

$$Q_N(x,y)=\begin{pmatrix} S_N^{(1)} & D_N^{(1)}\\ I_N^{(1)}-\epsilon & S_N^{(1)\dagger}\end{pmatrix}, \tag{4.11}$$

the correlation functions for ($\beta=1,N=$even) are given by

$$R_n(x_1,\ldots,x_n)=Q\,\det[Q_N(x_j,x_k)]_{j,k=1,\ldots,n}, \tag{4.12}$$

with Q det defined in the second step of Eq. (2.29).

Since $D_N^{(1)}(x,x)=I_N^{(1)}(x,x)=\epsilon(x,x)=0$, the level density is given by

$$R_1(x)=S_N^{(1)}(x,x). \tag{4.13}$$

For the universality, we need to prove that, with $(y-x)R_1(x)=r$,

$$\lim_{N\to\infty}\frac{S_N^{(1)}(x,y)}{S_N^{(1)}(x,x)}=S(r). \tag{4.14}$$

In that case, it would follow from Eqs. (4.8), (4.9) that

$$\lim_{N\to\infty}\frac{D_N^{(1)}(x,y)}{[S_N^{(1)}(x,x)]^2}=D(r), \tag{4.15}$$

and

$$\lim_{N\to\infty}I_N^{(1)}(x,y)=I(r), \tag{4.16}$$

giving thereby the unfolded correlation function $\mathbf{R}_n^{(1)}$ of Eqs. (2.29), (2.31). Note that Eqs. (4.13) and (4.14) are also valid when $S_N^{(1)}$ is replaced by $S_N^{(1)\dagger}$. Thus we would be concerned with Eqs. (4.13) and (4.14) only.

The skew-orthogonal polynomials appropriate in this case for the Jacobi weight function $w_{a,b}(x)$ of Eq. (2.9) are best described in terms of the Jacobi orthogonal polynomials $P_j^{2a+1,2b+1}(x)$ corresponding to the weight function $w_{2a+1,2b+1}(x)$. Note that

$$w_{a,b}(x)w_{a+1,b+1}(x)=w_{2a+1,2b+1}(x), \tag{4.17}$$

$$w_{a+1,b+1}(x)=(1-x^2)w_{a,b}(x), \tag{4.18}$$

the latter vanishing at $|x|=1$ for all $a>-1$, $b>-1$. These relations are useful in proving the results given below. Detailed proofs are given in Appendix A. We find that

$$\phi_{2m}(x)=w_{a,b}(x)P_{2m}^{2a+1,2b+1}(x), \tag{4.19}$$

$$\psi_{2m+1}(x)=w_{a+1,b+1}(x)P_{2m}^{2a+1,2b+1}(x), \tag{4.20}$$

with

$$g_{2m}=g_{2m+1}=h_{2m}^{2a+1,2b+1}. \tag{4.21}$$

The even-order ψ_{2m} and odd-order ϕ_{2m+1} are obtained by integrating Eq. (4.19) and differentiating Eq. (4.20), respectively:

$$\psi_{2m}(x)=\frac{1}{A_{2m}}w_{a+1,b+1}(x)P_{2m-1}^{2a+1,2b+1}(x) + \gamma_{2m-2}\psi_{2m-2}(x) \quad (m\neq 0), \tag{4.22}$$

$$\psi_0(x)=\int \epsilon(x-y)w_{a,b}(y)dy, \tag{4.23}$$

$$\phi_{2m+1}(x)=w_{a,b}(x)[A_{2m+1}P_{2m+1}^{2a+1,2b+1}(x) - B_{2m-1}P_{2m-1}^{2a+1,2b+1}(x)], \tag{4.24}$$

where

$$\gamma_j=\frac{(j+2a+2)(j+2b+2)}{(j+2)(j+2a+2b+4)}, \tag{4.25}$$

$$A_j=-\frac{j(j+2a+2b+2)}{(2j+2a+2b+1)}, \tag{4.26}$$

$$B_j=-\frac{(j+2a+2)(j+2b+2)}{(2j+2a+2b+5)}, \quad B_{-1}=0. \tag{4.27}$$

An important relation here is

$$\frac{d}{dx}\{w_{a+1,b+1}(x)P_j^{2a+1,2b+1}(x)\}=w_{a,b}(x) \times\{A_{j+1}P_{j+1}^{2a+1,2b+1}(x)-B_{j-1}P_{j-1}^{2a+1,2b+1}(x)\}, \tag{4.28}$$

which is helpful in obtaining Eqs. (4.19), (4.24) from Eqs. (4.22), (4.20). The normalization (4.21) is obtained by using Eqs. (4.19), (4.20) in Eq. (4.5). Note finally that the skew-orthogonal polynomials $q_j(x)$ are $\phi_j(x)/w_{a,b}(x)$. Mehta [8] has considered $a=b=0$, the Legendre case.

The asymptotic forms are derived by using Eq. (3.8) in the above results. Since $A_j\approx B_j\approx -j/2$ and $\gamma_j\approx 1$ for large j, we find, for large m,

$$(g_{2m})^{-1/2}\phi_{2m}(x)=\sqrt{\frac{2}{\pi\sin^3\theta}}\cos\left[\left(2m+a+b+\frac{3}{2}\right)\theta - \left(2a+\frac{3}{2}\right)\frac{\pi}{2}\right], \tag{4.29}$$

$$(g_{2m})^{-1/2}\psi_{2m+1}(x)=\sqrt{\frac{2\sin\theta}{\pi}}\cos\left[\left(2m+a+b+\frac{3}{2}\right)\theta - \left(2a+\frac{3}{2}\right)\frac{\pi}{2}\right], \tag{4.30}$$

$$(g_{2m})^{-1/2}\psi_{2m}(x)=-\frac{1}{m\sqrt{2\pi\sin\theta}}\sin\left[\left(2m+a+b+\frac{3}{2}\right)\theta - \left(2a+\frac{3}{2}\right)\frac{\pi}{2}\right], \tag{4.31}$$

$$(g_{2m})^{-1/2}\phi_{2m+1}(x)=2m\sqrt{\frac{2}{\pi\sin\theta}}\sin\left[\left(2m+a+b+\frac{3}{2}\right)\theta - \left(2a+\frac{3}{2}\right)\frac{\pi}{2}\right]. \tag{4.32}$$

Note that Eq. (4.31) is derived by partial integration of Eq. (4.29) to the leading order. However, there is an additional constant term (for $a\neq b$) of order $m^{-1/2}$, which obtains from the lower-order terms in the series in Eq. (4.22); this does not affect Eq. (4.34) below for $\Delta\theta=O(N^{-1})$ and is ignored. Thus we have

$$\frac{1}{g_{2m}}[\phi_{2m}(x)\psi_{2m+1}(y)-\phi_{2m+1}(x)\psi_{2m}(y)] = \frac{2\cos(2m\Delta\theta)}{\pi\sin\theta}, \tag{4.33}$$

so that

$$S_N^{(1)}(x,y)=\int_0^{N/2}\frac{2\cos(2m\Delta\theta)}{\pi\sin\theta}dm=\frac{\sin(N\Delta\theta)}{\pi\Delta\theta\sin\theta}. \tag{4.34}$$

With $\Delta\theta\rightarrow 0$, we obtain the level density (3.10) or (2.20) while, for $\Delta\theta=O(N^{-1})$, we obtain, as in Eq. (3.11), the result (4.14) for $\beta=1$. We have thus proved the universality of $\mathbf{R}_n^{(1)}$ for finite a,b and $|x|<1$.

The skew-orthogonal polynomials for the associated Laguerre weight function $w_a(x)$ of Eq. (2.11) are derived from the above Jacobi results and are given in terms of the associated Laguerre orthogonal polynomials $L_j^{(2a+1)}(2x)$ corresponding to the weight function $w_{2a+1}(2x)$; see Appendix A. Note here that Eqs. (4.17), (4.18) are now replaced by

$$2^{2a+1}w_a(x)w_{a+1}(x)=w_{2a+1}(2x), \tag{4.35}$$

$$w_{a+1}(x)=xw_a(x), \tag{4.36}$$

the latter vanishing at $x=0$ for all $a>-1$. We find now, instead of Eqs. (4.19)–(4.21),

$$\phi_{2m}(x)=2^{a+1/2}w_a(x)L_{2m}^{(2a+1)}(2x), \tag{4.37}$$

$$\psi_{2m+1}(x)=2^{a+3/2}w_{a+1}(x)L_{2m}^{(2a+1)}(2x), \tag{4.38}$$

with

$$g_{2m}=g_{2m+1}=h_{2m}^{(2a+1)}. \tag{4.39}$$

Similarly we find, instead of Eqs. (4.22)–(4.28),

$$\psi_{2m}(x)=\frac{2^{a+3/2}}{A_{2m}^L}w_{a+1}(x)L_{2m-1}^{(2a+1)}(2x) + \gamma_{2m-2}^L\psi_{2m-2}(x) \quad (m\neq 0), \tag{4.40}$$

$$\psi_0(x)=2^{a+1/2}\int \epsilon(x-y)w_a(y)dy, \tag{4.41}$$

$$\phi_{2m+1}(x)=2^{a+1/2}w_a(x)[A_{2m+1}^L L_{2m+1}^{(2a+1)}(2x) - B_{2m-1}^L L_{2m-1}^{(2a+1)}(2x)], \tag{4.42}$$

where

$$\gamma_j^L=\frac{(j+2a+2)}{(j+2)}, \tag{4.43}$$

$$A_j^L=j, \tag{4.44}$$

$$B_j^L=j+2a+2, \quad B_{-1}^L=0. \tag{4.45}$$

We also have

$$\frac{d}{dx}\{w_{a+1}(x)L_j^{(2a+1)}(2x)\}=\frac{1}{2}w_a(x)\{A_{j+1}^L L_{j+1}^{(2a+1)}(2x) - B_{j-1}^L L_{j-1}^{(2a+1)}(2x)\}, \tag{4.46}$$

again helpful in the derivation of Eqs. (4.40)–(4.42). With $x=(4m+2a+2)\cos^2\theta$, the asymptotic form for $L_j^{(2a+1)}(2x)$ is obtained from Eq. (3.14). Thus

$$(g_{2m})^{-1/2}\phi_{2m}(x)=\frac{1}{4m\sqrt{\pi\sin\theta\cos^3\theta}}\sin\left[(2m+a+1)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{4.47}$$

$$(g_{2m})^{-1/2}\psi_{2m+1}(x)=\frac{2}{\sqrt{\pi\tan\theta}}\sin\left[(2m+a+1)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{4.48}$$

$$(g_{2m})^{-1/2}\psi_{2m}(x)=-\frac{1}{4m\sqrt{\pi\sin^3\theta\cos\theta}}\cos\left[(2m+a+1)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{4.49}$$

$$(g_{2m})^{-1/2}\phi_{2m+1}(x)=2\sqrt{\frac{\tan\theta}{\pi}}\cos\left[(2m+a+1)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{4.50}$$

where, as in Eq. (4.31), the additional constant term of order $m^{-1/2}$ in Eq. (4.49) has been ignored. We have thus

$$\frac{1}{g_{2m}}[\phi_{2m}(x)\psi_{2m+1}(y)-\phi_{2m+1}(x)\psi_{2m}(y)] =\frac{\cos(8m\Delta\theta\sin^2\theta)}{2\pi m\sin\theta\cos\theta}=\frac{2\cos[x^{-1/2}(4m-x)^{1/2}\Delta x]}{\pi x^{1/2}(4m-x)^{1/2}}, \tag{4.51}$$

where in the last step $\Delta x=-8m\,\Delta\theta\sin\theta\cos\theta$. Using the last form of Eq. (4.51) (since θ varies with m but x does not), we find

$$S_N^{(1)}(x,y)=\int_{x/4}^{N/2}\frac{2\cos[x^{-1/2}(4m-x)^{1/2}\Delta x]}{\pi x^{1/2}(4m-x)^{1/2}}dm =\frac{\sin(x^{-1/2}(2N-x)^{1/2}\Delta x)}{\pi\Delta x}. \tag{4.52}$$

Again, $\Delta x\rightarrow 0$ gives

$$R_1(x)=\frac{1}{\pi}\sqrt{\frac{2N-x}{x}}, \quad 0<x<2N, \tag{4.53}$$

consistent with Eq. (2.24) for $\beta=1$, while $\Delta x R_1(x)=r$ with $N\rightarrow\infty$ gives the universal result (4.14).

In the Gaussian case, (2.10) with $\beta=1$, the Hermite polynomials $H_j(x)$ corresponding to $[w(x)]^2=\exp(-x^2)$ are again encountered. The skew-orthogonal polynomials can be either derived as the limiting case ($a=b\rightarrow\infty$) of the Jacobi results (see Appendix A) or read directly from the correlation-function results of Ref. [1]. We have

$$\phi_{2m}(x)=e^{-x^2/2}H_{2m}(x), \tag{4.54}$$

$$\psi_{2m+1}(x)=e^{-x^2/2}H_{2m}(x), \tag{4.55}$$

$$g_{2m}=g_{2m+1}=h_{2m}, \tag{4.56}$$

$$\psi_{2m}(x)=-2e^{-x^2/2}H_{2m-1}(x)+2(2m-1)\psi_{2m-2}(x) \quad (m\neq 0), \tag{4.57}$$

$$\psi_0(x)=\int_0^x e^{-y^2/2}H_0(y)dy, \tag{4.58}$$

$$\phi_{2m+1}(x)=e^{-x^2/2}[-(1/2)H_{2m+1}(x)+2mH_{2m-1}(x)]. \tag{4.59}$$

The corresponding asymptotic forms are obtained from Eq. (3.20). Using $x=(4m+1)^{1/2}\cos\theta$, we get Eq. (3.20) for $(g_{2m})^{-1/2}\phi_{2m}$ and $(g_{2m})^{-1/2}\psi_{2m+1}$ with $j=2m$, while for ψ_{2m} and ϕ_{2m+1} we have

$$(g_{2m})^{-1/2}\psi_{2m}(x)=-\frac{1}{2m^{3/4}\sqrt{\pi\sin^3\theta}}\cos\left[(m+1/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{4.60}$$

$$(g_{2m})^{-1/2}\phi_{2m+1}(x)=2m^{1/4}\sqrt{\frac{\sin\theta}{\pi}}\cos\left[(m+1/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right]. \tag{4.61}$$

There is no additional constant term in Eq. (4.60). Then, as in the Laguerre case,

$$S_N^{(1)}(x,y)=\int_{x^2/4}^{N/2}\frac{2\cos(\sqrt{(4m-x^2)}\Delta x)}{\pi\sqrt{4m-x^2}}dm=\frac{\sin(\sqrt{(2N-x^2)}\Delta x)}{\pi\Delta x}. \tag{4.62}$$

This gives the level density (2.28) when $\Delta x\to 0$ while $\Delta x R_1(x)=r$ with $N\to\infty$ gives the universal result (4.14).

V. JACOBI ORTHOGONAL ENSEMBLES (ODD DIMENSION)

When N is odd, the formal results of Sec. IV have to be modified [7] because the last skew-orthogonal polynomial is unpaired and its normalization is left arbitrary by Eq. (4.1). In this case we supplement Eq. (4.1) by the extra condition

$$\int q_j(x)w(x)dx=\delta_{j,N-1}, \tag{5.1}$$

and introduce the additional kernels

$$M(x,y)=\phi_{N-1}(x),\quad M^\dagger(x,y)=\phi_{N-1}(y), \tag{5.2}$$

$$\mu(x,y)=\psi_{N-1}(x),\quad \mu^\dagger(x,y)=\psi_{N-1}(y). \tag{5.3}$$

The result (4.12) is still valid if we define the quaternion Q for $\beta=1$, N odd, by

$$Q_N(x,y)=\begin{pmatrix} S_{N-1}^{(1)}+M & D_{N-1}^{(1)} \\ I_{N-1}^{(1)}-\epsilon+\mu-\mu^\dagger & S_{N-1}^{(1)\dagger}+M^\dagger \end{pmatrix}, \tag{5.4}$$

instead of Eq. (4.11).

Because of the extra condition (5.1) skew-orthogonal polynomials $q_j(x)$ of order j cannot be constructed. However, we can construct $q_j(x)$ of order $j+1$ for $j=0,\ldots,N-2$ and then construct $q_{N-1}(x)$ of order $N-1$; thus $q_{N-2}(x)$ and $q_{N-1}(x)$ will both be of order $(N-1)$ while there will be no polynomial of order 0.

For the Jacobi weight function (2.9), we have, with $m=0,1,\ldots,(N-3)/2$,

$$\phi_{2m}(x)=w_{a,b}(x)[P_{2m+1}^{2a+1,2b+1}(x)-c_m], \tag{5.5}$$

$$c_m=\int w_{a,b}(y)P_{2m+1}^{2a+1,2b+1}(y)dy, \tag{5.6}$$

$$\psi_{2m+1}(x)=w_{a+1,b+1}(x)P_{2m+1}^{2a+1,2b+1}(x), \tag{5.7}$$

while

$$g_{2m}=g_{2m+1}=h_{2m+1}^{2a+1,2b+1}, \tag{5.8}$$

and

$$\phi_{N-1}(x)=\frac{w_{a,b}(x)P_{N-1}^{2a+1,2b+1}(x)}{\int w_{a,b}(y)P_{N-1}^{2a+1,2b+1}(y)dy}. \tag{5.9}$$

To prove these results, note first that, without the constant c_m, Eqs. (5.5), (5.7) satisfy the skew orthogonality as in Eqs. (4.19), (4.20) of the preceding section. The constant c_m ensures that the condition (5.1) is satisfied for $\phi_{2m}(x)$ without affecting the skew-orthogonality with the other functions. The condition (5.1) is automatically satisfied for $\phi_{2m+1}(x)$, because $\psi_{2m+1}(1)=0$. Finally, $\phi_{N-1}(x)$ of Eq. (5.9) is skew orthogonal with the other ϕ's and is normalized according to Eq. (5.1). The results for $\phi_{2m+1}(x)$ and $\psi_{2m}(x)$ can be worked out along the steps followed in Eqs. (4.22)–(4.28). For the asymptotic forms, Eqs. (4.29)–(4.32) apply with the replacement $m\to m+1/2$ in the phases of the main cosine and sine terms. Note that the additional c_m term in Eq. (5.5) does not contribute to the asymptotic forms of the kernel function, nor do the additional terms involving μ and M. We have thus the same level density and the universal unfolded correlation functions as in the even-N case.

For the associated Laguerre case, we have Eqs. (4.37)–(4.39) with $2m\to 2m+1$ on the right hand sides and the c_m term of the type (5.5),(5.6) for $\phi_{2m}(x)$. For $\phi_{N-1}(x)$, we have Eq. (5.9) with $P_{N-1}^{2a+1,2b+1}\to L_{N-1}^{(2a+1)}$. The asymptotic forms (4.47)–(4.50) again apply with the replacement $m\to m+1/2$ in the phases.

For the Gaussian case, we again have Eqs. (4.54)–(4.56) with $2m\to 2m+1$ on the right-hand side. In this case the c_m correction of the type (5.6) is not needed, as $c_m=0$. With $P_{N-1}\to H_{N-1}$ in Eq. (5.9) we get $\phi_{N-1}(x)$. These results are identical with the Gaussian-ensemble results of Ref. [1].

VI. JACOBI SYMPLECTIC ENSEMBLES

When $\beta=4$, we require a different family of skew-orthogonal polynomials $t_j(x)$, defined by

$$\int [t_j(x)t_k'(x)-t_k(x)t_j'(x)]w(x)dx=g_jZ_{jk}. \quad (6.1)$$

[Our definition of $t_j(x)$ is related [8] to the definition in Ref. [7] by a derivative operation]. Here g_j is again a normalization constant satisfying $g_{2m}=g_{2m+1}$. The t_j polynomials can be chosen to be of order j with $j=0,1,2,\ldots$, and have other properties similar to that of q_j discussed in Sec. IV. It will be useful to write Eq. (6.1) as

$$\int [\phi_j(x)\phi_k'(x)-\phi_k(x)\phi_j'(x)]dx=g_jZ_{jk}, \quad (6.2)$$

where

$$\phi_j(x)=[w(x)]^{1/2}t_j(x). \quad (6.3)$$

To write the correlation functions, we define the kernels, analogous to Eqs. (4.6)–(4.9), by

$$S_{2N}^{(4)}(x,y)=-\sum_{j,k=0}^{2N-1}(g_j)^{-1}Z_{jk}\phi_j'(x)\phi_k(y), \quad (6.4)$$

$$S_{2N}^{(4)\dagger}(x,y)=S_{2N}^{(4)}(y,x), \quad (6.5)$$

$$D_{2N}^{(4)}(x,y)=\sum_{j,k=0}^{2N-1}(g_j)^{-1}Z_{jk}\phi_j'(x)\phi_k'(y)=-\frac{\partial S_{2N}^{(4)}(x,y)}{\partial y}, \quad (6.6)$$

$$I_{2N}^{(4)}(x,y)=-\sum_{j,k=0}^{2N-1}(g_j)^{-1}Z_{jk}\phi_j(x)\phi_k(y)=\int_y^x S_{2N}^{(4)}(z,y)dz. \quad (6.7)$$

The correlation functions for $\beta=4$ are then given by Eq. (4.12) with

$$Q_N(x,y)=\begin{pmatrix} S_{2N}^{(4)}(x,y) & D_{2N}^{(4)}(x,y)\\ I_{2N}^{(4)}(x,y) & S_{2N}^{(4)\dagger}(x,y)\end{pmatrix}. \quad (6.8)$$

We remark that the kernels (6.4)–(6.7) are somewhat different from those given in Refs. [7,8], but the correlation-function determinants are the same. Our choice is convenient for asymptotic studies below. The level density is given by

$$R_1(x)=S_{2N}^{(4)}(x,x), \quad (6.9)$$

as in Eq. (4.13) for $\beta=1$. With $\Delta x R_1(x)=r$, we need to prove that

$$\lim_{N\to\infty}\frac{S_{2N}^{(4)}(x,y)}{S_{2N}^{(4)}(x,x)}=S(2r), \quad (6.10)$$

$$\lim_{N\to\infty}\frac{D_{2N}^{(4)}(x,y)}{S_{2N}^{(4)}(x,x)}=D(2r), \quad (6.11)$$

$$\lim_{N\to\infty}\frac{I_{2N}^{(4)}(x,y)}{S_{2N}^{(4)}(x,x)}=I(2r), \quad (6.12)$$

to obtain the universal $\mathbf{R}_n$ of Eqs. (2.29), (2.32) for $\beta=4$. Note also that, as for $\beta=1$, Eqs. (6.9), (6.10) remain valid with $S_{2N}^{(4)}$ replaced by $S_{2N}^{(4)\dagger}$. Since Eqs. (6.11), (6.12) follow from Eq. (6.10), we would again be concerned only with Eqs. (6.9), (6.10).

We consider first the Jacobi weight function (2.9). As shown in Appendix A, the $t_j'(x)$ can be written compactly in terms of the Jacobi orthogonal polynomials $P_j^{a,b}(x)$,

$$t_{2m+1}'(x)=P_{2m}^{a,b}(x), \quad (6.13)$$

$$t_{2m}'(x)=P_{2m-1}^{a,b}(x)+\eta_{2m}t_{2m-2}'(x), \quad (6.14)$$

where η_{2m} is a constant, given in Eq. (6.21) below. On integration, we find the polynomials:

$$t_{2m+1}(x)=\frac{2}{(2m+a+b)}[D_{2m+1}P_{2m+1}^{a,b}(x)+E_{2m+1}P_{2m}^{a,b}(x)+F_{2m+1}P_{2m-1}^{a,b}(x)], \quad (6.15)$$

$$t_{2m}(x)=\frac{2}{(2m+a+b-1)}[D_{2m}P_{2m}^{a,b}(x)+E_{2m}P_{2m-1}^{a,b}(x)+F_{2m}P_{2m-2}^{a,b}(x)]+\eta_{2m}t_{2m-2}(x). \quad (6.16)$$

Equations (6.13)–(6.16) are valid for $m=0,1,2,\ldots$, if we take $t_j(x)$ and $P_j^{a,b}(x)$ as zero for negative j. In Eqs. (6.15), (6.16) we have used the indefinite integral

$$\frac{1}{2}(j+a+b)\int P_j^{a,b}(x)dx=P_{j+1}^{a-1,b-1}(x)=D_{j+1}P_{j+1}^{a,b}(x)+E_{j+1}P_j^{a,b}(x)+F_{j+1}P_{j-1}^{a,b}(x). \quad (6.17)$$

The integration constants have been put equal to zero because of skew orthogonality with $t_1(x)$. The constants D_j, E_j, F_j, η_j, and g_j are given by

$$D_j=\frac{(j+a+b)(j+a+b-1)}{(2j+a+b)(2j+a+b-1)}, \quad (6.18)$$

$$E_j=\frac{(a-b)(j+a+b-1)}{(2j+a+b)(2j+a+b-2)}, \quad (6.19)$$

$$F_j=-\frac{(j+a-1)(j+b-1)}{(2j+a+b-1)(2j+a+b-2)}, \quad (6.20)$$

$$\eta_j=\frac{(j+a-1)(j+b-1)(2j+a+b-5)}{(j-1)(j+a+b-1)(2j+a+b-1)}, \quad (6.21)$$

$$g_{2m}=g_{2m+1}=\frac{2h_{2m}^{a,b}}{4m+a+b-1}=\frac{2^{a+b+2}\Gamma(2m+a+1)\Gamma(2m+b+1)}{(4m+a+b+1)(4m+a+b-1)\Gamma(2m+1)\Gamma(2m+a+b+1)}. \tag{6.22}$$

For large j and large m,

$$D_j=-F_j=\frac{1}{4}+O(j^{-1}),\quad E_j=O(j^{-1}), \tag{6.23}$$

$$\eta_j=1+O(j^{-1}),\quad g_{2m}=\frac{2^{a+b}}{4m^2}+O(m^{-3}), \tag{6.24}$$

and, in the same approximation,

$$t_{2m+1}(x)=\frac{1}{4m}[P_{2m+1}^{a,b}(x)-P_{2m-1}^{a,b}(x)], \tag{6.25}$$

$$t_{2m}(x)=\frac{1}{4m}\{P_{2m}^{a,b}(x)+2^{(a+b)/2}[w_{a,b}(x)]^{-1/2}\}. \tag{6.26}$$

Here in Eq. (6.26) the nonpolynomial term on the right-hand side is the large-m approximation for the lower-order terms in the series in Eq. (6.16) and has been verified numerically. Then, with $x=\cos\theta$ and using Eq. (3.8), we find the leading terms in the large-m expansion of the polynomials,

$$(g_{2m})^{-1/2}\phi_{2m+1}(x)=-\sqrt{\frac{\sin\theta}{\pi m}}\sin\left[\left(2m+\frac{a+b+1}{2}\right)\theta-\left(a+\frac{1}{2}\right)\frac{\pi}{2}\right], \tag{6.27}$$

$$(g_{2m})^{-1/2}\phi_{2m}(x)=\frac{1}{2}\left\{\frac{1}{\sqrt{\pi m\sin\theta}}\cos\left[\left(2m+\frac{a+b+1}{2}\right)\theta-\left(a+\frac{1}{2}\right)\frac{\pi}{2}\right]+1\right\}, \tag{6.28}$$

$$(g_{2m})^{-1/2}\phi_{2m+1}'(x)=2\sqrt{\frac{m}{\pi\sin\theta}}\cos\left[\left(2m+\frac{a+b+1}{2}\right)\theta-\left(a+\frac{1}{2}\right)\frac{\pi}{2}\right], \tag{6.29}$$

$$(g_{2m})^{-1/2}\phi_{2m}'(x)=\sqrt{\frac{m}{\pi\sin^3\theta}}\sin\left[\left(2m+\frac{a+b+1}{2}\right)\theta-\left(a+\frac{1}{2}\right)\frac{\pi}{2}\right], \tag{6.30}$$

where Eqs. (6.29), (6.30) are obtained by a partial differentiation of Eqs. (6.27), (6.28). With $\Delta\theta$ as defined before,

$$\frac{1}{g_{2m}}[\phi_{2m}(y)\phi_{2m+1}'(x)-\phi_{2m+1}(y)\phi_{2m}'(x)]=\frac{\cos(2m\Delta\theta)}{\pi\sin\theta}, \tag{6.31}$$

where we have ignored the rapidly oscillating term arising from the constant in Eq. (6.28). Hence

$$S_{2N}^{(4)}(x,y)=\int_0^N\frac{\cos(2m\Delta\theta)}{\pi\sin\theta}dm=\frac{\sin(2N\Delta\theta)}{2\pi\Delta\theta\sin\theta}. \tag{6.32}$$

Again, we get, on appropriate limits, the level density (3.10) from Eq. (6.9), and (6.10) for the kernel, confirming the stationarity and universality for $\beta=4$.

For the associated Laguerre weight function (2.11), we obtain from Eqs. (6.13)–(6.16), after suitable limits (see Appendix A),

$$t_{2m+1}'(x)=L_{2m}^{(a)}(x), \tag{6.33}$$

$$t_{2m}'(x)=L_{2m-1}^{(a)}(x)+\left(\frac{2m+a-1}{2m-1}\right)t_{2m-2}'(x), \tag{6.34}$$

$$t_{2m+1}(x)=-L_{2m+1}^{(a)}(x)+L_{2m}^{(a)}(x), \tag{6.35}$$

$$t_{2m}(x)=-L_{2m}^{(a)}(x)+L_{2m-1}^{(a)}(x)+\left(\frac{2m+a-1}{2m-1}\right)t_{2m-2}(x). \tag{6.36}$$

For $a=0$, Eqs. (6.35), (6.36) give back the results of Ref. [8], with the observation that any multiple of $t_{2m}(x)$ can be added to $t_{2m+1}(x)$. The normalization constant is given by

$$g_{2m}=g_{2m+1}=-h_{2m}^{(a)}. \tag{6.37}$$

The results (6.35),(6.36) derive from Eqs. (6.33), (6.34) from the indefinite integral,

$$\int L_j^{(a)}(x)dx=-L_{j+1}^{(a-1)}(x)=-L_{j+1}^{(a)}(x)+L_j^{(a)}(x), \tag{6.38}$$

the constants of integration in Eqs. (6.35), (6.36) being zero on skew orthogonality with $t_1(x)$. With $x=(8m+2a+4)\cos^2\theta$ corresponding effectively to $j=2m+\frac{1}{2}$ in Eq. (3.14), the asymptotic forms are given by

$$|g_{2m}|^{-1/2}\phi_{2m+1}(x)=\frac{1}{\sqrt{\pi m\tan\theta}}\sin\left[(2m+1+a/2)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \tag{6.39}$$

$$|g_{2m}|^{-1/2}\phi_{2m}(x) = -\frac{1}{2}\left\{\frac{1}{2\sqrt{\pi m}\cos\theta\sin^3\theta}\cos\left[(2m+1+a/2)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right]+1\right\}, \quad (6.40)$$

$$|g_{2m}|^{-1/2}\phi'_{2m+1}(x)=\frac{1}{2}\sqrt{\frac{\tan\theta}{\pi m}}\cos\left[(2m+1+a/2)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \quad (6.41)$$

$$|g_{2m}|^{-1/2}\phi'_{2m}(x)=\frac{1}{8\sqrt{\pi m}\cos^3\theta\sin\theta}\sin\left[(2m+1+a/2)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \quad (6.42)$$

where we have used $|g_{2m}|$, since g_{2m} is negative. In deriving Eq. (6.40) we have used the large-m approximation

$$\left(\frac{8m-x}{2m}\right)t_{2m}(x)=-[L_{2m}^{(a)}(x)+L_{2m+1}^{(a)}(x)]-\frac{1}{2}\left(\frac{8m-x}{2m}\right)\times(2m)^{a/2}[w_a(x)]^{-1/2}, \quad (6.43)$$

which follows from the three-term recursion and a sum rule for $L_j^{(a)}(x)$ [16]. Ignoring, as in the Jacobi case, the rapidly oscillating term, we obtain

$$\frac{1}{g_{2m}}[\phi_{2m}(y)\phi'_{2m+1}(x)-\phi_{2m+1}(y)\phi'_{2m}(x)] = \frac{\cos\left[\frac{1}{2}x^{-1/2}(8m-x)^{1/2}\Delta x\right]}{\pi x^{1/2}(8m-x)^{1/2}}, \quad (6.44)$$

so that

$$S_{2N}^{(4)}(x,y)=\int_{x/8}^{N}\frac{\cos\left[\frac{1}{2}x^{-1/2}(8m-x)^{1/2}\Delta x\right]}{\pi x^{1/2}(8m-x)^{1/2}}dm = \frac{\sin\left[\frac{1}{2}x^{-1/2}(8N-x)^{1/2}\Delta x\right]}{2\pi\Delta x}. \quad (6.45)$$

Again, with $\Delta x\to 0$, we get the level density (2.24) corresponding to $\beta=4$, while the limit for finite r gives the universal result (6.10).

We finally turn to the Gaussian case. We have now, with $w(x)=\exp(-2x^2)$,

$$t'_{2m+1}(x)=2(2m+1)H_{2m}(x\sqrt{2}), \quad (6.46)$$

$$t'_{2m}(x)=4mH_{2m-1}(x\sqrt{2})+4mt'_{2m-2}(x), \quad (6.47)$$

$$t_{2m+1}(x)=\frac{1}{\sqrt{2}}H_{2m+1}(x\sqrt{2}), \quad (6.48)$$

$$t_{2m}(x)=\frac{1}{\sqrt{2}}H_{2m}(x\sqrt{2})+4mt_{2m-2}(x), \quad (6.49)$$

along with the normalization

$$g_{2m}=g_{2m+1}=(2m+1)!\,\pi^{1/2}2^{2m}. \quad (6.50)$$

For large m, we have, with $x=(2m+3/2)^{1/2}\cos\theta$ corresponding to $j=2m+1$ and $x\to x\sqrt{2}$ in Eq. (3.20),

$$(g_{2m})^{-1/2}\phi_{2m+1}(x)=\frac{1}{m^{1/4}\sqrt{\pi\sin\theta}}\sin\left[(m+3/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \quad (6.51)$$

$$(g_{2m})^{-1/2}\phi_{2m}(x)=\frac{1}{(4m)^{1/4}}\left\{\frac{1}{2\sqrt{2m\pi\sin^3\theta}}\cos\left[(m+3/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right]+\frac{1}{2}\right\}, \quad (6.52)$$

$$(g_{2m})^{-1/2}\phi'_{2m+1}(x)=2m^{1/4}\sqrt{\frac{2\sin\theta}{\pi}}\cos\left[(m+3/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right], \quad (6.53)$$

$$(g_{2m})^{-1/2}\phi'_{2m}(x)=-\frac{1}{(m)^{1/4}\sqrt{2\pi\sin\theta}}\sin\left[(m+3/4)\times(\sin 2\theta-2\theta)+\frac{3\pi}{4}\right]. \quad (6.54)$$

Here, in deriving Eq. (6.52), we have used the large-m approximation,

$$2\sqrt{2}t_{2m}(x)=\frac{1}{4(2m-x^2)}[-H_{2m+2}(x\sqrt{2})+4mH_{2m}(x\sqrt{2})]+m^{-1/4}(g_{2m})^{1/2}e^{x^2}. \quad (6.55)$$

Now, using Eqs. (6.51)–(6.54) in $S_{2N}^{(4)}$, we have,

$$S_{2N}^{(4)}(x,y)=\int_{x^2/2}^{N}\frac{\cos(2\sqrt{(2m-x^2)}\Delta x)}{\pi\sqrt{2m-x^2}}dm = \frac{\sin(2\sqrt{(2N-x^2)}\Delta x)}{2\pi\Delta x}, \quad (6.56)$$

so that we get the level density (2.28) with $\Delta x \to 0$, and the universal kernel (6.10) for finite r.

VII. ASYMPTOTIC RESULTS FOR GENERAL WEIGHT FUNCTIONS

We consider in this section a general procedure for obtaining asymptotic forms of the polynomials without going through the tedious process of deriving their exact forms. Our approach is similar to that of Brezin and Zee [14] who considered the $\beta=2$ case for weight functions of the type $w(x)=\exp[-NV(x)]$ where $V(x)$ is a low-order polynomial. We consider all weight functions for which the level density $R_1(x)$ can be derived for large N from Eq. (2.14), and the polynomials of all three types. Our asymptotic results will be in conformity with those of Secs. IV, V, and VI and hence the universality will follow for a wider class of weight functions. We also give exact integral representations of the polynomials, extending thereby the $\beta=2$ result of Szego (Ref. [16] Chap. 2) and Eynard [12], from which the asymptotics can be worked out rigorously.

We start with the $\beta=2$ case. It is convenient to deal with the orthonormal set of functions

$$\phi_j(x)=(h_j)^{-1/2}[w(x)]^{1/2}p_j(x), \tag{7.1}$$

defined in terms of the orthogonal polynomials $p_j(x)$ of Sec. III. We propose the ansatz that, for asymptotic j, $\phi_j(x)$ can be written as

$$\phi_j(x)=A_j(x)\cos[\Theta_j(x)+\chi(x)], \tag{7.2}$$

where the amplitude $A_j(x)$ is a slowly varying function of j, the phase $\Theta_j(x)$ grows indefinitely with increasing j, and the phase $\chi(x)$ is either independent of j or slowly varying with j. Using Eq. (7.2) in Eq. (3.5), replacing summation by an integral, and ignoring the rapidly oscillating contribution to the integral, we find, for large N,

$$R_1(x,N)=\frac{1}{2}\int_0^N [A_j(x)]^2 dj, \tag{7.3}$$

giving thereby the amplitude in terms of the level density $R_1(x,j)$ of the j dimensional ensemble,

$$[A_j(x)]^2=2\frac{\partial R_1(x,j)}{\partial j}. \tag{7.4}$$

In the asymptotic case, R_1 is also the density of zeros of the polynomials. This is implicit in Chap. 6 of Ref. [16] in the Jacobi case, but is observed to be true more generally from the integral representations discussed later. The spacing D $(=R_1^{-1})$ between the consecutive zeros of ϕ_j is given by

$$\Theta_j(x+D)-\Theta_j(x)=D\frac{\partial \Theta_j}{\partial x}=\pi, \tag{7.5}$$

where we have assumed that $D(x)|\chi'(x)|\ll 1$. Thus

$$\Theta_j(x)=\pi\int_c^x R_1(x',j)dx'. \tag{7.6}$$

Here [c, d] is the support of the density function R_1. If we use d instead of c, the difference would be $j\pi$, changing thereby only the sign of the polynomial. Any other choice of the lower limit will make $\chi(x)$ j- dependent. With Eqs. (7.2), (7.4), the orthonormality of the $\phi_j(x)$ is easily verified,

$$\begin{aligned}\int \phi_j(x)\phi_k(x)dx &= \frac{1}{2}\int_c^d dx A_j(x)A_k(x)\Bigg\{\cos\Bigg[\pi\int_c^x[R_1(x',j)\\ &\quad -R_1(x',k)]dx'\Bigg]+\cos\Bigg[\pi\int_c^x[R_1(x',j)\\ &\quad +R_1(x',k)]dx'+2\chi(x)\Bigg]\Bigg\}\\ &=\int_c^d dx\frac{\partial R_1(x,j)}{\partial j}\cos\Bigg[(j-k)\pi\int_c^x\left(\frac{\partial R_1(x',j)}{\partial j}\right)dx'\Bigg]\\ &=\delta_{jk},\end{aligned} \tag{7.7}$$

where both the terms in the first step vanish for j,k large and far apart, while the second step is for finite $|j-k|$. One can also verify the three-term recursion relations asymptotically. Our final result (7.2) along with Eqs. (7.4) and (2.14) leaves $\chi(x)$ undetermined, but is consistent with the asymptotic results for Jacobi, associated Laguerre and Hermite polynomials of Sec. III as well as with the Brezin and Zee ansatz for $w(x)=\exp[-NV(x)]$. In the latter case note that $N^{-1}R_1(x,N)$ becomes independent of N as is evident from (2.14). Chapter 12 of Ref. [16] discusses other generalizations that are also consistent with our results.

Taking $R_1=0$ for $x<c$ and $x>d$, we can write the ansatz in the form,

$$\phi_j(x)=\left(\frac{2\,\partial R_1(x,j)}{\partial j}\right)^{1/2}\cos\left[\pi\int_{-\infty}^x R_1(x',j)dx'+\chi(x)\right], \tag{7.8}$$

suitable for comparison with the ansatz for $\beta=1,4$ below. Here, without loss of generality, the sign of $A_j(x)$ in Eq. (7.8) has been taken to be positive. Moreover, $R_1(x,j)$ follows from Eq. (2.14) with $\beta=2$ and $\chi(x)$ satisfies

$$\left|\frac{\partial\chi}{\partial j}\right|\ll 1,\quad \frac{1}{R_1}\left|\frac{\partial\chi}{\partial x}\right|\ll 1. \tag{7.9}$$

We believe that both quantities in Eq. (7.9) are of order j^{-1}. Now, with summation replaced by an integral, the kernel $S_N^{(2)}$ of Eq. (3.2) for large N is given by

$$S_N^{(2)}(x,x+\Delta x)=\int_0^N \phi_j(x)\phi_j(x+\Delta x)dj$$
$$=\int_0^N \frac{\partial R_1(x,j)}{\partial j}\cos(\pi R_1(x,j)\Delta x)dj$$
$$=\frac{\sin(\pi R_1(x,N)\Delta x)}{\pi\Delta x}, \tag{7.10}$$

so that the universal result,

$$\lim_{N\to\infty}\frac{S_N^{(2)}(x,x+rD(x))}{S_N^{(2)}(x,x)}=\frac{\sin \pi r}{\pi r}, \tag{7.11}$$

is obtained for a wide class of weight functions.

For the skew-orthogonal polynomials of the $\beta=1$ type, we redefine $\phi_j(x)$ of Eq. (4.3) as

$$\phi_j(x)=(g_j)^{-1/2}w(x)q_j(x)=\frac{d\psi_j(x)}{dx}. \tag{7.12}$$

Then, along with $\psi_k(x)$ of Eq. (4.4), we have the skew-orthonormality relation,

$$\int \phi_j(x)\psi_k(x)dx=Z_{jk}. \tag{7.13}$$

Now, we propose the ansatz

$$\phi_{2m}(x)=\pi B_m(x)R_1(x,2m)\times\cos\left[\pi\int_{-\infty}^{x}R_1(x',2m)dx'+\xi(x)\right], \tag{7.14}$$

$$\psi_{2m+1}(x)=[\pi B_m(x)R_1(x,2m)]^{-1}\times\frac{\partial R_1(x,2m)}{\partial m}\cos\left[\pi\int_{-\infty}^{x}R_1(x',2m)dx'+\xi(x)\right], \tag{7.15}$$

$$\psi_{2m}(x)=B_m(x)\sin\left[\pi\int_{-\infty}^{x}R_1(x',2m)dx'+\xi(x)\right], \tag{7.16}$$

$$\phi_{2m+1}(x)=-[B_m(x)]^{-1}\frac{\partial R_1(x,2m)}{\partial m}\times\sin\left[\pi\int_{-\infty}^{x}R_1(x',2m)dx'+\xi(x)\right], \tag{7.17}$$

where R_1 is given by Eq. (2.14) with $\beta=1$, but B_m and ξ, as χ above, are undetermined. Equations (7.14)–(7.17) are valid for the even-dimensional case of Sec. IV. For the odd-dimensional case of Sec. V, the formal expressions are the same with the replacement $R_1(x,2m)\to R_1(x,2m+1)$. Proof of these results are similar to the $\beta=2$ case, with orthonormality replaced by skew orthonormality and the kernel $S_N^{(2)}$ replaced by the kernel $S_N^{(1)}$ of Eq. (4.6). In the even-dimensional case (the odd-dimensional case is handled similarly), we first write the ansatz in the form,

$$\phi_{2m}(x)=A_{2m}(x)\cos[\Theta_{2m}(x)+\xi_1(x)], \tag{7.18}$$

$$\phi_{2m+1}(x)=A_{2m+1}(x)\cos[\bar{\Theta}_{2m}(x)+\xi_2(x)], \tag{7.19}$$

allowing for different forms for the even and odd-order polynomials. The integral representations below will prove that the zeros of $\phi_{2m}(x)$ have the density $R_1(x,2m)$, giving thereby the integrated density in the phase in Eq. (7.14). Similarly zeros of $\phi_{2m+1}(x)$ have the same density $R_1(x,2m)$, except for the density of an additional zero that is absorbed in $\xi_2(x)$; thus $\bar{\Theta}_{2m}(x)=\Theta_{2m}(x)$ as in Eq. (7.17). The integrated functions $\psi_{2m}(x)$ and $\psi_{2m+1}(x)$ are obtained by partial integrations, giving *inter alia* an additional factor $[\pi R_1(x,2m)]^{-1}$ in the amplitudes. Skew orthogonality gives $\xi_1(x)=\xi_2(x)$, while evaluation of $R_1(x)=S_N^{(1)}(x,x)$ gives $A_{2m}(x)A_{2m+1}(x)=\pi R_1(x,2m)\partial R_1(x,2m)/\partial m$. Writing $A_{2m}(x)=\pi B_m(x)R_1(x,m)$, we get Eqs. (7.14)–(7.17) with $\xi(x)$ satisfying the conditions (7.9) for $\chi(x)$. The skew normality is automatically satisfied to the leading order. The kernel $S_N^{(1)}$ is now given by

$$S_N^{(1)}(x,x+\Delta x)=\int_0^{N/2}[\phi_{2m}(x)\psi_{2m+1}(x+\Delta x)-\phi_{2m+1}(x)\psi_{2m}(x+\Delta x)]dm$$
$$=\int_0^{N/2}\frac{\partial R_1(x,2m)}{\partial m}\cos(\pi R_1(x,2m)\Delta x)dm$$
$$=\frac{\sin(\pi R_1(x,N)\Delta x)}{\pi\Delta x}, \tag{7.20}$$

which, on taking $N\to\infty$ limit, gives the universal result (4.14) for $\beta=1$.

The asymptotic forms for the $\beta=4$ type weighted skew-orthonormal polynomials $\phi_j(x)$ are given by

$$\phi_{2m}(x)=C_m(x)\cos\left[2\pi\int_{-\infty}^{x}R_1(x',m)dx'+\zeta(x)\right], \tag{7.21}$$

$$\phi_{2m+1}(x)=[2\pi C_m(x)R_1(x,m)]^{-1}\times\frac{\partial R_1(x,m)}{\partial m}\sin\left[2\pi\int_{-\infty}^{x}R_1(x',m)dx'+\zeta(x)\right], \tag{7.22}$$

$$\phi'_{2m}(x) = -2\pi C_m(x) R_1(x,m) \times \sin\left[2\pi \int_{-\infty}^{x} R_1(x',m)dx' + \zeta(x)\right], \tag{7.23}$$

$$\phi'_{2m+1}(x) = [C_m(x)]^{-1} \frac{\partial R_1(x,m)}{\partial m} \times \cos\left[2\pi \int_{-\infty}^{x} R_1(x',m)dx' + \zeta(x)\right], \tag{7.24}$$

where $R_1(x,m)$ is given by Eq. (2.14) with $\beta=4$ while $C_m(x)$ and $\zeta(x)$ are undetermined with $\zeta(x)$ satisfying the conditions (7.9) for $\eta(x)$. We have used the definition $\phi_j(x) = (g_j)^{-1/2}[w(x)]^{1/2} t_j(x)$ instead of Eq. (6.3) with unit normalization in Eq. (6.2). The proof of Eqs. (7.21)–(7.24) is almost the same as that of Eqs. (7.14)–(7.17). Starting with the ansatz in Eqs. (7.18), (7.19), we make the following changes. The integral representations below will prove that, for $\beta=4$, ϕ_{2m}, and ϕ_{2m+1} have doubly degenerate zeros (except for one extra zero in ϕ_{2m+1}) so that we get integrals of $2R_1(x,m)$ instead of $R_1(x,2m)$ for $\Theta_{2m} = \overline{\Theta}_{2m}$. We have additional factors of R_1 in ϕ'_j (instead of R_1^{-1} as in ψ_j), since partial differentiation is used. From $R_1(x) = S_{2N}^{(4)}(x,x)$, we get $A_{2m}(x) A_{2m+1}(x) = [2\pi R_1(x,m)]^{-1} \partial R_1(x,m)/\partial m$. Writing $A_{2m}(x) = C_m(x)$ gives then Eqs. (7.21)–(7.24). The kernel $S_{2N}^{(4)}$ of Eq. (6.4) is now given by

$$S_{2N}^{(4)}(x, x+\Delta x) = \frac{\sin(2\pi R_1(x,N)\Delta x)}{2\pi \Delta x}, \tag{7.25}$$

which gives, for $\beta=4$, the universal result (6.10) in the limit.

We have assumed here (as well as for $\beta=1$ above) that the g_j are positive. When the g_j are negative, $|g_j|^{-1/2}$ should be used in the definition of ϕ_j with change of sign in either ϕ_{2m} or ϕ_{2m+1} (as in the $\beta=4$ case of the associated Laguerre weight function). Moreover, since $g_{2m} = g_{2m+1}$, alternative expressions for the skew-orthonormal functions $\phi_j(x)$ involving other powers of the g_j are possible. The ansatz (7.14)–(7.17) and Eqs. (7.21)–(7.24) are consistent with the asymptotic results given in Secs. IV, VI, respectively, for $\beta = 1,4$ for the entire Jacobi family of weight functions. For these weight functions, the B_m and C_m are easily determined,

$$B_m(x) \propto [R_1(x,2m)]^{-1/2} \frac{\partial R_1(x,2m)}{\partial m}, \tag{7.26}$$

$$C_m(x) \propto [R_1(x,m)]^{-1/2} \frac{\partial R_1(x,m)}{\partial m}, \tag{7.27}$$

the proportionality constant being $(4m)^{-1/2}$, 2^{-1}, $2^{-1/2}$ for $B_m(x)$ and 2^{-1}, $m^{1/2}$, $2^{-5/4}$ for $C_m(x)$, respectively, for the Jacobi, associated Laguerre, and Hermite cases.

As discussed in Secs. IV, VI, Eqs. (7.16), (7.21) have additional constant terms that contribute rapidly oscillating terms in Eqs. (7.20), (7.25) without affecting the final universal results. For $\beta=4$, the extra term in Eq. (7.21) arises because, as discussed below, the even-order polynomials have complex zeros whose imaginary parts become small for large m; in this case $2R_1(x,m)$ is the density of the real parts of the zeros.

We now turn to the integral representations for the three types of polynomials. These are given as averages over matrix ensembles, and will be useful in deriving the above asymptotic forms rigorously. We define the average of F with respect to the eigenvalue distribution (2.2) for given β, N as

$$\langle F(x_1, \ldots, x_N) \rangle_{\beta,N} = \int \ldots \int F(x_1, \ldots, x_N) \times \mathcal{P}_{\beta,N}(x_1, \ldots, x_N) dx_1 \ldots dx_N. \tag{7.28}$$

We consider monic polynomials, i.e., polynomials with highest coefficient unity. Then the orthogonal polynomials, appropriate for $\beta=2$, are given [16,12] by

$$p_j(x) = \left\langle \prod_{k=1}^{j} (x - x_k) \right\rangle_{2,j}. \tag{7.29}$$

The skew-orthogonal polynomials, appropriate for $\beta=1$ (even-dimensional case), are given by

$$q_{2m}(x) = \left\langle \prod_{k=1}^{2m} (x - x_k) \right\rangle_{1,2m}, \tag{7.30}$$

$$q_{2m+1}(x) = \left\langle \left(x + \sum_{k=1}^{2m} x_k \right) \prod_{k=1}^{2m} (x - x_k) \right\rangle_{1,2m}. \tag{7.31}$$

For $\beta=1$ (odd-dimensional case), we have

$$q_{2m}(x) = \left\langle \prod_{k=1}^{2m+1} (x - x_k) \right\rangle_{1,2m+1} - C_m, \tag{7.32}$$

$$q_{2m+1}(x) = \left\langle \left(x + \sum_{k=1}^{2m+1} x_k \right) \prod_{k=1}^{2m+1} (x - x_k) \right\rangle_{1,2m+1}, \tag{7.33}$$

valid for $m = 0,1, \ldots, (N-3)/2$. Here C_m is determined from the condition (5.1). q_{N-1} is given (in the nonmonic form) by

$$q_{N-1}(x) = \frac{\left\langle \prod_{k=1}^{N-1} (x - x_k) \right\rangle_{1,N-1}}{\int dx\, w(x) \left\langle \prod_{k=1}^{N-1} (x - x_k) \right\rangle_{1,N-1}}. \tag{7.34}$$

For $\beta=4$, the skew-orthogonal polynomials are

$$t_{2m}(x) = \left\langle \prod_{k=1}^{m} (x - x_k)^2 \right\rangle_{4,2m}, \tag{7.35}$$

$$t_{2m+1}(x)=\left\langle \left(x+2\sum_{k=1}^{m} x_k\right)\prod_{k=1}^{m}(x-x_k)^2\right\rangle_{4,2m}. \tag{7.36}$$

A proof of these integral representations is outlined in Appendix B. Note that the above averages for all three β can be written in terms of $\det(x-H)$ and $(\mathrm{tr}\,H)\det(x-H)$ averaged over the matrix ensemble of H. We mention moreover that, in a recent unpublished work, Eynard [13] has also obtained Eqs. (7.30), (7.31), (7.35), and (7.36).

We finally give a heuristic proof, needed in the above ansatz, of the relation between the density of zeros of the polynomials and the level density of the corresponding ensembles. It is known that the orthogonal polynomials $p_j(x)$ have real zeros [16]. Our numerical studies indicate that the zeros of the skew-orthogonal polynomials $q_{2m}(x), q_{2m+1}(x)$ and $t_{2m+1}(x)$ are also real, while those of the $t_{2m}(x)$ are complex, having small imaginary parts for large m and hence doubly degenerate real parts. To find the density of zeros, we take the eigenvalue spectrum $\{x_k\}$ to be ordered (i.e., $x_1 \leqslant x_2 \leqslant x_3 \ldots$) and write $x_k=\langle x_k\rangle+\delta x_k$. The spectrum of the average eigenvalues $\{\langle x_k\rangle\}$ has asymptotically the density $R_1(x)$. The fluctuations δx_k are small [2], spanning over a few spacings [i.e., $\langle \delta x_k \delta x_l\rangle \approx (R_1(x))^{-2}$ for k,l not too far apart]; the δx_k then can be ignored in the leading approximations in Eqs. (7.29)–(7.36). Thus, for $\beta=2$, $p_j(x) \simeq \Pi(x-\langle x_j\rangle)$, giving thereby the zeros as $\langle x_j\rangle$ with the density $R_1(x,j)$. For $\beta=1$, we have similarly the density $R_1(x,2m)$ for the zeros of $q_{2m}(x)$ with an additional term for the zero at $\Sigma\langle x_k\rangle$ for $q_{2m+1}(x)$. For $\beta=4$, the leading approximations for $t_{2m}(x)$ in Eq. (7.35) gives doubly degenerate zeros at $\langle x_k\rangle$ with density $2R_1(x,m)$, while $t_{2m+1}(x)$ has an additional term corresponding to the zero at $(-2\Sigma\langle x_k\rangle)$. Since $t_{2m}(x)$ is positive definite, the zeros of $t_{2m}(x)$ must be complex with small imaginary parts for large m.

VIII. CONCLUSION

The universality of energy-level fluctuations, observed in a wide range of physical systems, was first considered by Fox and Kahn [11] for the unitary ensembles and later extended by Dyson [7] to the orthogonal and symplectic ensembles. Dyson conjectured that "the local statistical properties of the eigenvalues become universal properties independent of the global eigenvalue distributions" in the limit of large dimensionality. We have established the universality rigorously for the Jacobi class of weight functions and via an ansatz for more general weight functions. We have also shown that the local properties are stationary, being independent of the location in the spectrum. Our proof relies on a systematic study of the skew-orthogonal polynomials and their asymptotic forms for the Jacobi class (including the associated Laguerre and Gaussian cases). For the more general weight funtions our ansatz for the asymptotic polynomials relies on a heuristic derivation of the density of zeros of the polynomials. The matrix-integral representations of the polynomials—orthogonal as well as skew orthogonal ones—appear to be promising for rigorous studies in the general case. We have also given a formalism for deriving the (nonuniversal) level density without using the polynomial method.

We believe that the concept of skew orthogonality will be useful for other random-matrix systems and in other contexts. The local correlation functions have universal properties also for the Brownian-motion matrix ensembles [7,17], viz., ensembles interpolating between the invariant ones. These ensembles are useful in the studies of small symmetry breaking in quantum chaotic systems [18]. In the GOE-GUE [23] and GSE-GUE [24] interpolations, the concept of skew orthogonality has been implicitly used. Similarly polynomials on the unit circle are used in the study of circular ensembles [17,19,21]. The skew-orthogonal polynomials may be useful in the study of other Brownian-motion ensembles. We also believe that new methods of semiclassical quantization [25] of chaotic systems can be developed using skew-orthogonal functions as tools.

ACKNOWLEDGMENTS

We thank Sanjay Puri for valuable discussions.

APPENDIX A: PROOF FOR JACOBI SKEW-ORTHOGONAL POLYNOMIALS

In this appendix we prove the Jacobi results for the skew-orthogonal polynomials given in Secs. IV, VI for $\beta=1,4$, respectively.

We start with $\beta=1$. Since $q_j(x)$ is a polynomial of order j, the weighted polynomials $\phi_j(x)=w_{a,b}(x)q_j(x)$ can be written, without loss of generality, as

$$\phi_j(x)=\gamma_j^{(j)}w_{a,b}(x)\{A_jP_j^{2a+1,2b+1}(x)-B_{j-2}P_{j-2}^{2a+1,2b+1}(x)\}+\sum_{k=0}^{j-1}\gamma_k^{(j)}\phi_k(x), \tag{A1}$$

valid for $j\geqslant 1$, $\phi_0(x)$ being $w_{a,b}(x)$. The $\gamma_k^{(j)}$ are the expansion coefficients, and the A_j, B_j are given in Eqs. (4.26)–(4.27). We choose

$$\gamma_{2m}^{(2m)}=(A_{2m})^{-1}, \quad \gamma_{2m+1}^{(2m+1)}=1, \tag{A2}$$

to fix the leading coefficient in the $q_j(x)$. Then using Eqs. (4.4), (4.28), we have

$$\psi_j(x)=\gamma_j^{(j)}w_{a+1,b+1}(x)P_{j-1}^{2a+1,2b+1}(x)+\sum_{k=0}^{j-1}\gamma_k^{(j)}\psi_k(x), \tag{A3}$$

valid for $j\geqslant 1$, while $\psi_0(x)$ is given by Eq. (4.23). From the orthogonality of the Jacobi polynomials we have

$$\int w_{a+1,b+1}(x)P_j^{2a+1,2b+1}(x)\phi_k(x)dx = \int w_{2a+1,2b+1}(x)P_j^{2a+1,2b+1}(x)q_k(x)dx=0, \tag{A4}$$

for $k=0,\ldots,j-1$. Using this and the skew-orthogonality relation (4.5), we find that

$$\gamma_k^{(2m+2)}=0, \tag{A5}$$

for $k\neq 2m,2m+2$, and

$$\gamma_k^{(2m+1)}=0, \tag{A6}$$

$k\neq 2m,2m+1$. Moreover, since $\gamma_{2m}^{(2m+1)}$ is arbitrary (as skew orthogonality is not affected), we choose it to be zero and then Eq. (A6) is valid for $k=2m$ also. Thus $\gamma_{2m}^{(2m+2)} \equiv\gamma_{2m}$ is nonzero, giving thereby Eqs. (4.20), (4.22). The skew orthogonality of $q_{2m+1}(x)$ and $q_{2m+2}(x)$ gives

$$\gamma_{2m}=\frac{h_{2m+1}^{2a+1,2b+1}}{h_{2m-1}^{2a+1,2b+1}}\frac{A_{2m+1}}{A_{2m+2}}\frac{A_{2m}}{B_{2m-1}}=\frac{B_{2m}}{A_{2m+2}}, \tag{A7}$$

and hence Eq. (4.25), the last step in Eq. (A7) following from Eq. (A9) below. Using the $\gamma_k^{(j)}$ in Eq. (A1) gives the $\phi_j(x)$ in Eqs. (4.19), (4.24) and then the normalization (4.21) is obtained from Eqs. (4.19), (4.20). Finally, the relation (4.28) is proved from

$$\int_{-1}^{1}dx w_{a+1,b+1}(x)P_j^{2a+1,2b+1}(x)\frac{d}{dx}[w_{a+1,b+1}(x)P_k^{2a+1,2b+1}(x)] = -\int_{-1}^{1}dx w_{a+1,b+1}(x)P_k^{2a+1,2b+1}(x)\frac{d}{dx}[w_{a+1,b+1}(x)P_j^{2a+1,2b+1}(x)], \tag{A8}$$

which is nonzero only for $|j-k|=1$. Now, doing the integrals in Eqs. (A8) for $|j-k|=1$ in terms of $h_j^{2a+1,2b+1}$ and $k_j^{2a+1,2b+1}$, we find

$$A_j=-(j+a+b+1)\frac{k_{j-1}^{2a+1,2b+1}}{k_j^{2a+1,2b+1}},\quad B_j=\frac{h_{j+1}^{2a+1,2b+1}}{h_j^{2a+1,2b+1}}A_{j+1}, \tag{A9}$$

and hence Eqs. (4.26), (4.27). This completes the proof of all the results (4.19)–(4.28).

For associated Laguerre and Hermite weight functions one can directly follow the above procedure, or more simply take the limits of the Jacobi results as discussed below. For associated Laguerre, note first that

$$w_a(x)=\lim_{b\to\infty}2^{-a-b}b^a w_{ab}(1-2b^{-1}x), \tag{A10}$$

$$L_j^{(a)}(x)=\lim_{b\to\infty}P_j^{a,b}(1-2b^{-1}x), \tag{A11}$$

$$k_j^{(a)}=\lim_{b\to\infty}\left(-\frac{2}{b}\right)^j k_j^{a,b}, \tag{A12}$$

$$h_j^{(a)}=\lim_{b\to\infty}\frac{b^{a+1}}{2^{a+b+1}}h_j^{a,b}. \tag{A13}$$

Thus for skew-orthogonal functions we have (in terms of the Jacobi skew-orthogonal functions $\phi_j^{a,b},\psi_j^{a,b}$),

$$\phi_j(x)=\lim_{b\to\infty}(-1)^j 2^{-b+1/2}b^a\phi_j^{a,b}(1-2b^{-1}x), \tag{A14}$$

$$\psi_j(x)=\lim_{b\to\infty}(-1)^{j-1}2^{-b-1/2}b^{a+1}\psi_j^{a,b}(1-2b^{-1}x), \tag{A15}$$

giving thereby Eqs. (4.37)–(4.46). Similarly, for the Hermite case, note that (with $j=2m,2m+1$)

$$e^{-x^2/2}=\lim_{a\to\infty}w_{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{A16}$$

$$H_j(x)=\lim_{a\to\infty}2^j j!\,a^{-j/2}P_j^{a,a}\left(\frac{x}{\sqrt{a}}\right), \tag{A17}$$

$$k_j=\lim_{a\to\infty}2^j j!\,a^{-j}k_j^{a,a}, \tag{A18}$$

$$h_j=\lim_{a\to\infty}(2^j j!)^2 a^{-j+1/2}h_j^{a,a}, \tag{A19}$$

$$\phi_j(x)=\lim_{a\to\infty}2^{2m}(2m)!(2a)^{-j/2}\phi_j^{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{A20}$$

$$\psi_j(x)=\lim_{a\to\infty}2^{2m}(2m)!(2a)^{-(j-1)/2}\psi_j^{a,a}\left(\frac{x}{\sqrt{2a}}\right), \tag{A21}$$

giving the Hermite results (4.54)–(4.59).

For $\beta=4$ with the Jacobi weight function, we expand $t_j'(x)$ as

$$t_j'(x)=P_{j-1}^{a,b}(x)+\sum_{k=0}^{j-1}\eta_k^{(j)}t_k'(x), \tag{A22}$$

so that Eq. (6.17) gives

$$t_j(x)=\frac{2}{j+a+b-1}[D_jP_j^{a,b}(x)+E_jP_{j-1}^{a,b}(x)+F_jP_{j-2}^{a,b}(x)] +\sum_{k=0}^{j-1}\eta_k^{(j)}t_k(x). \tag{A23}$$

Then the orthogonality of the $P_j^{a,b}$ and the skew orthogonality of the t_j give

$$\eta_k^{(2m)}=0, \tag{A24}$$

for $k\neq 2m-2$, and

$$\eta_k^{(2m+1)}=0, \tag{A25}$$

for $k\neq 2m$. Also $\eta_{2m}^{(2m+1)}$, being arbitrary, is chosen to be zero. Thus $\eta_{2m-2}^{(2m)}\equiv\eta_{2m}$ is the only nonzero coefficient, giving thereby Eqs. (6.13)–(6.16). The skew orthogonality of t_{2m} and t_{2m-1} gives

$$\eta_{2m}=\frac{1}{g_{2m-2}}\left[\frac{2D_{2m-1}}{2m+a+b-2}h_{2m-1}^{a,b} -\frac{2F_{2m}}{2m+a+b-1}h_{2m-2}^{a,b}\right], \tag{A26}$$

while the normalization is given by

$$g_{2m}=\left[\frac{2D_{2m}}{2m+a+b-1}h_{2m}^{a,b}-\frac{2F_{2m+1}}{2m+a+b}h_{2m-1}^{a,b}\right], \tag{A27}$$

confirming thereby Eqs. (6.21), (6.22). To prove the Jacobi result (6.17), we note that the first step is given in a differential form in Refs. [16,26], while for the second step we use [26]

$$(2j+a+b)P_j^{a,b-1}(x)=(j+a+b)P_j^{a,b}(x) +(j+a)P_{j-1}^{a,b}(x), \tag{A28}$$

$$(2j+a+b)P_j^{a-1,b}(x)=(j+a+b)P_j^{a,b}(x)-(j+b)P_{j-1}^{a,b}(x). \tag{A29}$$

This completes the proof of Eqs. (6.13)–(6.22).

The associated Laguerre results (6.33)–(6.38) derive directly by using the limits (A10)–(A13) in Eqs. (6.13)–(6.22), while the Hermite results (6.46)–(6.49) derive from the limits

$$e^{-2x^2}=\lim_{a\to\infty}w_{a,a}(x\sqrt{2/a}), \tag{A30}$$

$$H_j(x\sqrt{2})=\lim_{a\to\infty}2^j j!\,a^{-j/2}P_j^{a,a}(x\sqrt{2/a}). \tag{A31}$$

APPENDIX B: PROOF OF MATRIX-INTEGRAL REPRESENTATIONS

In this appendix we outline a proof of the matrix-integral representations (7.29)–(7.36) of the polynomials. The Vandermonde determinant (2.3) and its fourth power can be written as [1],

$$\Delta_N(x_1,\ldots,x_N)=\det[x_\mu^{N-\nu}]_{\mu,\nu=1,\ldots,N}, \tag{B1}$$

$$[\Delta_N(x_1,\ldots,x_N)]^4=\det[x_\mu^{2N-\nu},(2N-\nu)x_\mu^{2N-\nu-1}]_{\mu=1,\ldots,N,\nu=1,\ldots,2N}. \tag{B2}$$

For $\beta=2$, Eq. (7.29) represents orthogonal polynomials with the weight $w(x)$, if

$$\int x^k p_j(x)w(x)dx=0, \tag{B3}$$

for $k=0,1,\ldots,j-1$. Using Eqs. (2.2), (7.28), and (7.29) along with Eqs. (2.3), (B1), we find that the integral in Eq. (B3) is proportional [12] to

$$\int dx_1\ldots\int dx_{j+1}(x_{j+1})^k\Delta_j(x_1,\ldots,x_j)\Delta_{j+1}(x_1,\ldots,x_{j+1})\prod_{\mu=1}^{j+1}w(x_\mu)$$
$$=\frac{1}{(j+1)!}\int dx_1\ldots\int dx_{j+1}\left(\sum_P\epsilon_P(x_{i_{j+1}})^k\Delta_j(x_{i_1},\ldots,x_{i_j})\right)\Delta_{j+1}(x_1,\ldots,x_{j+1})\prod_{\mu=1}^{j+1}w(x_\mu), \tag{B4}$$

where Σ_P is a summation over all permutations $(x_{i_1},\ldots,x_{i_{j+1}})$ of $(x_1,\ldots,x_{j+1})$ and $\epsilon_P(=\pm 1)$ is the sign of the permutation, equal to the change of sign in Δ_{j+1} after the permutation. The summation term in Eq. (B4) can be written as

$$\sum_P \epsilon_P (x_{i_{j+1}})^k \Delta_j(x_{i_1},\ldots,x_{i_j}) = (-1)^j \det\begin{pmatrix} x_1^k & x_2^k & \ldots & x_{j+1}^k \\ x_1^{j-1} & x_2^{j-1} & \ldots & x_{j+1}^{j-1} \\ . & . & . & . \\ .. & . & . & . \\ 1 & 1 & \ldots & 1 \end{pmatrix}, \tag{B5}$$

which is zero for $k=0,1,\ldots,j-1$, proving thereby Eq. (B3) and hence Eq. (7.29).

For $\beta=1$, we consider Eqs. (7.30), (7.31) for the even-N case; a similar consideration would apply to Eqs. (7.32)–(7.34) for the odd-N case. The q_j of Eqs. (7.30), (7.31) represent skew-orthogonal polynomials of the $\beta=1$ type with the weight $w(x)$, if

$$\int\int dx\,dy\,\epsilon(x-y)y^k q_j(x)w(x)w(y)=0, \tag{B6}$$

for $k=0,\ldots,2m-1$ and also for $k=j$ for $j=2m,2m+1$ both. The integrals in Eqs. (7.30), (7.31) involve $|\Delta_N(x_1,\ldots,x_{2m})|$ and therefore Mehta's method of integration [1] over alternate variables can be used. For $j=2m$, the integral in Eq. (B6) is proportional to

$$\begin{aligned}
&\int dx_1\ldots dx_{2m+2}\,\epsilon(x_{2m+1}-x_{2m+2})(x_{2m+2})^k\left(\prod_{\nu=1}^{2m}(x_{2m+1}-x_\nu)\right)|\Delta_{2m}(x_1,\ldots,x_{2m})|\prod_{\mu=1}^{2m+2}w(x_\mu)\\
&=(2m)!\int_{x_1\leqslant x_2\leqslant\ldots\leqslant x_{2m}}dx_1\ldots\int dx_{2m+2}\,\epsilon(x_{2m+1}-x_{2m+2})(x_{2m+2})^k\left(\prod_{\mu=1}^{2m+2}w(x_\mu)\right)\Delta_{2m+1}(x_1,\ldots,x_{2m+1})\\
&=\frac{1}{2}\,\frac{(2m)!}{m!}\int dx_1dx_3\ldots dx_{2m+1}\det\begin{pmatrix} 0 & 0 & \ldots & x_{2m+1}^k & F_k(x_{2m+1})\\ x_1^{2m} & F_{2m}(x_1) & \ldots & x_{2m}^{2m} & F_{2m}(x_{2m+1})\\ . & . & . & . & .\\ . & . & . & . & .\\ 1 & F_0(x_1) & \ldots & 1 & F_0(x_{2m+1})\end{pmatrix}\prod_{i=0}^{m}w(x_{2i+1})\\
&=\frac{1}{2}\,\frac{(2m)!}{m!(m+1)!}\int dx_1dx_3\ldots dx_{2m+1}\det\begin{pmatrix} x_1^k & F_k(x_1) & \ldots\\ x_1^{2m} & F_{2m}(x_1) & \ldots\\ x_1^{2m-1} & F_{2m-1}(x_1) & \ldots\\ . & . & \ldots.\\ . & . & \ldots.\\ 1 & F_0(x_1) & \ldots\end{pmatrix}\prod_{i=0}^{m}w(x_{2i+1}),
\end{aligned}\tag{B7}$$

where in the second and third steps the above-mentioned Mehta's method of integration over alternate variables is used and $F_k(x)$ is given by

$$F_k(x)=\int_x^\infty y^k w(y)dy. \tag{B8}$$

In the last step of Eq. (B7) all permutations of $(x_1,x_3,\ldots,x_{2m+1})$ have been used. The determinant in the last step is zero for $k=0,\ldots,2m$, proving thereby Eqs. (B6) and (7.30). For $j=2m+1$, the first integral in Eq. (B7) has, in the integrand, the extra factor $(x_{2m+1}+\Sigma x_\nu)$ so that Δ_{2m+1} in the second form is replaced by

$$\det\begin{pmatrix} x_1^{2m+1} & \dots & x_{2m+1}^{2m+1} \\ x_1^{2m-1} & \dots & x_{2m+1}^{2m-1} \\ . & \dots & . \\ . & \dots & . \\ 1 & \dots & 1 \end{pmatrix} = \left(\sum_{\mu=1}^{2m+1} x_\mu \right) \Delta_{2m+1}(x_1, \dots, x_{2m+1}). \tag{B9}$$

Then the second row in the determinant of the last step of (B7) is replaced by $x_1^{2m+1}, F_{2m+1}(x_1), \dots,$ other rows remaining the same. Again Eq. (B6) for $k=0, \dots, 2m-1, 2m+1$ and hence Eq. (7.31) are verified.

For $\beta=4$, Eqs. (7.35)–(7.36) represent the skew-orthogonal polynomials if

$$\int dx \{x^k t_j'(x) - k x^{k-1} t_j(x)\} w(x) = 0, \tag{B10}$$

for $k=0, \dots, 2m-1$ and also for $k=j$ for $j=2m, \quad 2m+1$ both. In this case we use Eq. (B2) in the joint-probability density (2.2). For $j=2m$, the integral in Eq. (B10) is proportional to

$$\int dx_1 \dots dx_{2m} dx_{2m+1} \left\{ x_{2m+1}^k \frac{d}{dx_{2m+1}} \prod_{\nu=1}^{m} (x_{2m+1}-x_\nu)^2 - k x_{2m+1}^{k-1} \prod_{\nu=1}^{m} (x_{2m+1}-x_\nu)^2 \right\} \left(\prod_{\mu=1}^{2m+1} w(x_\mu) \right) [\Delta_m(x_1, \dots, x_m)]^4$$

$$= \int dx_1 \dots dx_{2m+1} \det\begin{pmatrix} 0 & 0 & \dots & x_{2m+1}^k & k x_{2m+1}^{k-1} \\ x_1^{2m} & 2m x_1^{2m-1} & \dots & x_{2m+1}^{2m} & 2m x_{2m+1}^{2m-1} \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 1 & 0 & \dots & 1 & 0 \end{pmatrix} \prod_{\mu=1}^{2m+1} w(x_\mu)$$

$$= \frac{1}{(m+1)!} \int dx_1 \dots dx_{2m+1} \det\begin{pmatrix} x_1^k & k x_1^{k-1} & \dots \\ x_1^{2m} & 2m x_1^{2m-1} & \dots \\ . & . & \dots \\ . & . & \dots \\ 1 & 0 & \dots \end{pmatrix} \prod_{\mu=1}^{2m+1} w(x_\mu), \tag{B11}$$

where the last step is by a permutation of all the variables in the first step. The determinant in the last step is again zero for $k=0, \dots, 2m$, confirming Eq. (B10) and hence Eq. (7.35). For $j=2m+1$, we have the additional term $(x_{2m+1}+2\Sigma x_\nu)$ with $\Pi(x-x_\nu)^2$. In this case the second row of both the determinants of Eq. (B11) are replaced by $[x_1^{2m+1}, (2m+1)x_1^{2m}, \dots,]$, the last determinant being then zero for $k=0, \dots, 2m-1$, and $2m+1$. Thus Eq. (7.36) is verified.

---

[1] M. L. Mehta, *Random Matrices* (Academic, New York, 1991).

[2] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).

[3] O. Bohigas and M. J. Giannoni, Lect. Notes Phys. **209**, 1 (Springer, Berlin, 1984).

[4] T. Guhr, A. M. Groeling, and H. A. Weidenmueller, Phys. Rep. **299**, 189 (1998).

[5] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).

[6] F. Haake, *Quantum Signatures Of Chaos* (Springer, Berlin, 2001).

[7] F. J. Dyson, J. Math. Phys. **13**, 90 (1972).

[8] M. L. Mehta, *Matrix Theory* (Hindustan Publishing Corporation, Delhi, 1989).

[9] A. Pandey and S. Ghosh, Phys. Rev. Lett. **87**, 024102 (2001).

[10] R. Balian, Nuovo Cimento B **57**, 183 (1968).

[11] D. Fox and P. B. Kahn, Phys. Rev. **134**, B1151 (1964).

[12] B. Eynard, Nucl. Phys. B **506**, 633 (1997).

[13] B. Eynard, Saclay Report No. SPT-00/170, CRM 2698 (unpublished).

[14] E. Brezin and A. Zee, Nucl. Phys. B **402**, 613 (1993).

[15] A. Pandey, Ann. Phys. (N.Y.) **119**, 170 (1979).

[16] G. Szego, *Orthogonal Polynomials* (American Mathematical Society, Providence, 1939).

[17] A. Pandey, Chaos Solitons Fractals **5**, 1275 (1995).

[18] J. B. French, V. K .B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 198 (1988).

[19] A. Pandey and P. Shukla, J. Phys. A **24**, 3907 (1991).

[20] A. Pandey, Ann. Phys. (N.Y.) **134**, 110 (1981).

[21] F. J. Dyson, Commun. Math. Phys. **19**, 235 (1970).
[22] M. L. Mehta, Commun. Math. Phys. **20**, 245 (1971).
[23] A. Pandey and M. L. Mehta, Commun. Math. Phys. **87**, 449 (1983).
[24] M. L. Mehta and A. Pandey, J. Phys. A **16**, 2655 (1983).
[25] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).
[26] M. Abramowitz and I. A. Stegan, *Handbook of Mathematical Functions* (Dover, New York, 1970).